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TRAVEL TIMES AND RAY PATHS FOR CONTINUOUS MEDIA

J. A. MADRID*
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RESUMEN

Las observaciones de tiempos de recorrido y amplitudes son de gran importancia en la interpretación sismológica tanto de fuente como de estructura. Se presenta aquí un método nuevo parà obtener tiempos de recorrido y rayos para medios cuya velocidad de propagación de onda puede ser especificada analíticamente. Se proveen ejemplos en dos dimensiones y se desarrolla una extensión a tres dimensiones. Los resultados de aplicarlo a un modelo, se pueden usar en la construcción de sismogramas sintéticos o en esquemas de inversión de tiempos e interpretación. Numéricamente, este método es una extensión de otros métodos para resolver ecuaciones diferenciales; la técnica usual de predictor - corrector se reemplaza por un criterio simple. Computacionalmente es sencillo, barato y arroja resultados precisos.

ABSTRACT

Travel times and amplitude observations are of prime importance in seismic interpretation of both source and structure. A new method for computing travel times and ray paths for media whose velocity law of wave propagation and boundaries are specified analitically, is presented here. The method is illustrated by several examples in two dimensions, and an extension to the three dimensional case is developed. This method facilitates further computations related to the construction of synthetic seismograms, and provides better tools for data inversion and model interpretation. Numerically, this method is an improvement over existing methods for solving certain types of differential equations, where a ray path is considered as a curve in space. Computationally, the method is straightforward, and yields accurate results with relatively little effort. The usual predictor-corrector control is replaced by a pair of easy to apply criteria.

* *Centro de Investigación Científica y Educación Superior de Ensenada, (CICESE), B. Cfa., MEXICO.*

1. INTRODUCTION

Travel times and amplitudes of observed records are powerful tools in seismic data analysis. Until very recently, seismic media have usually been modeled by layers where velocity of propagation depends in simple ways on depth only. New methods have been reported for computing travel time curves and ray paths and for inverting travel time data in laterally heterogeneous media. Some of these methods solve a system of partial differential equations numerically, while others divide the media in regions where either the velocity or its gradient are homogeneous, and use previously known analytical solutions of the ray problem. Examples may be found in Pereyra, Lee and Keller, (1980), Lentini and Pereyra (1977), Jacob (1970), Marks and Hron (1980), Will (1976), Whittal and Clowes (1979), Julian and Gubbins (1977), Green (1976), Aric *et al.* (1980). Pure numerical solutions are expensive, and the use of regions produce singularities that affect later calculations (synthetic seismograms, for example).

A major difficulty of the inverse problem is the lack of uniqueness of solutions. Generally, the number of possible solutions is reduced by geological restrictions and, if possible, by comparisons of observed and computed amplitudes. This step requires generating synthetic seismograms. There are many solutions to this problem, the most important or realistic ones being very difficult to use. Complete methods yield solutions as difficult to interpret as the actual recorded data (Alterman and Karal, 1968; Smith, 1975) and simpler methods that provide intermediate results suffer from singularities (Cagniard de - Hoop, Reflectivity, Asymptotic Ray Theory). In general, calculations are lengthy and expensive, except in Asymptotic Ray Theory (ART), which is valid for heterogeneous media.

Despite its simplicity there are singular cases in ART, for instance, velocity discontinuities produce triplications, and gradient discontinuities cause caustics. This peculiarities may introduce difficulties in the computation of synthetic seismograms even if the gradient discontinuities are artificial and the triplications are small. In a recent paper by Chapman and Drummond (1982), this inconvenience is not important because the WKBJ algorithm automatically smooths the results. In all these methods the computation of the travel times and ray paths is crucial, because they reflect characteristics of the medium.

In this paper, I propose a simple method for computing travel times and ray paths for media whose velocities are continuous functions of the coordinates, except at actual (physical) discontinuities. The method is a limiting case of the "circular approximation" (Marks and Hron, 1980; Aric *et al.*, 1980), as well as an analogy of Euler's method for solving differential equations of the first order. Gebrande (1976) reported a method essentially similar, but the basic formulas used to compute the ray displacement differ, and here a three dimensional procedure is explained.

A ray is a curve representing the evolution of a specific point on a wavefront from the instant it is produced until it emerges to the free surface. At a given instant, the path describes locally an arc of a circle contained in an instantaneous plane of propagation. This plane contains the vector of slowness and of gradient of velocity. The local circle is well known in differential geometry as the "oscullating circle", an important characteristic being that its first and second derivatives coincide with the corresponding derivatives of the ray path curve (Eisenhart, 1909, 1960). In the next section the method will be described and justified, examples will be given and the results will be compared with the results of other authors. Extension to three dimensions will be discussed in section 3.

The method does not require of large computer space. Models are specified with few parameters. The final result is an interactive program which provides a mean to readily search for regions of interest and more realistic ray paths.

2. DESCRIPTION OF THE METHOD

Several methods for seismic ray tracing and for travel time determination have been reported recently. Most of them divide the medium under study into regions whose velocity or velocity gradient are homogeneous, e.g. the circular approximation (Marks and Hron, 1980; Aric *et al.*, 1980). Chapman and Drummond (1982) have used this technique to obtain data for the computation of body wave seismograms using Maslov's Asymptotic Theory. In that paper, the authors remark that, in order to obtain significant amplitudes, the velocity must be a smooth function of the coordinates of the medium, so as to prevent the the formation of spurious critical points, caustics, and triplications. The medium is divided in triangular regions (two dimensions). Each region is characterized by a linear velocity law so that travel time and geometrical spreading are easy to find. The gradient discontinuities are small and do not cause too much trouble in forming the synthetic seismogram. Real discontinuities are made to coincide with appropriate sides of triangles, and the velocity does not change discontinuously in sides which are not real boundaries.

This work originates in the circular approximation. The idea is to divide a medium in a very large number of very small triangular regions. If the propagation velocity of a wave is represented through the medium as a smooth function of the coordinates, every point in the medium becomes equivalent to an infinitesimal triangular region so that the velocity may be approximated by a Taylor expansion of first order, thus making it possible to apply the formula of the circular approximation in a neighborhood of each point. In this way, forced use of discontinuities of either the velocity or its gradient, and the formation of spurious critical points and caustics, may be prevented.

Suppose a ray through a seismic medium for which the variation of velocity is small. We can write

$$v = v(x, z) \qquad (1)$$

in a neighborhood of (x_0, z_0):

$$\begin{aligned} v(x, z) &= vo + (\partial v/\partial x)_0\,(x - xo) + (\partial v/\partial z)o(z - zo) \\ &= vo' = b_1 x + b_2 z \end{aligned} \qquad (2)$$

where

$$b_1 = \partial v/\partial x, \qquad b_2 = \partial v/\partial z$$

and

$$b = (b_1^2 + b_2^2)^{1/2} \qquad (3)$$

is the absolute value of the gradient vector.

The small triangle in Fig. 1 illustrates this case. Inside, the gradient vector is constant and forms an angle

$$r = -\tan^{-1}(b_1/b_2) \qquad (4)$$

with the negative z-axis. It is well known (Nettleton, 1940) that in a medium of constant velocity gradient, the ray describes an arc of a circle whose radius is given by

$$R = 1/p'b \tag{5}$$

where $p' = \text{sen}\, i'/v$ is the horizontal slowness in a system of coordinates rotated by an angle r (Fig. 2), so that the direction of the z'-axis coincides with the direction of $-\underline{b}$. In the rotated system (x', z'), called "local system" from now on, the velocity may be expressed by:

$$v(x, z) = v_0' - bz' \tag{6}$$

so that p' is constant. The local slowness components are obtained from the "external" slowness components (p, q) by the usual rotation in two dimensions:

$$\begin{aligned} p' &= p \cos r + q \sin r \\ q' &= -p \sin r + q \cos r \end{aligned} \tag{7}$$

where $\cos r = -b_2/b$ and $\text{sen}\, r = b_1/b$; combining these with (7) and (5), it is possible to write

$$R = (-pb_2 + qb_1) \tag{8}$$

for the radius of curvature of the ray at a point. The center of the local circle is located at:

$$\begin{aligned} Xc &= xo + R \sin\theta \\ Zc &= zo + R \cos\theta \end{aligned} \tag{9}$$

And the circle satisfies the equation

$$(x - Xc)^2 + (z - Zc)^2 = R^2 \tag{10}$$

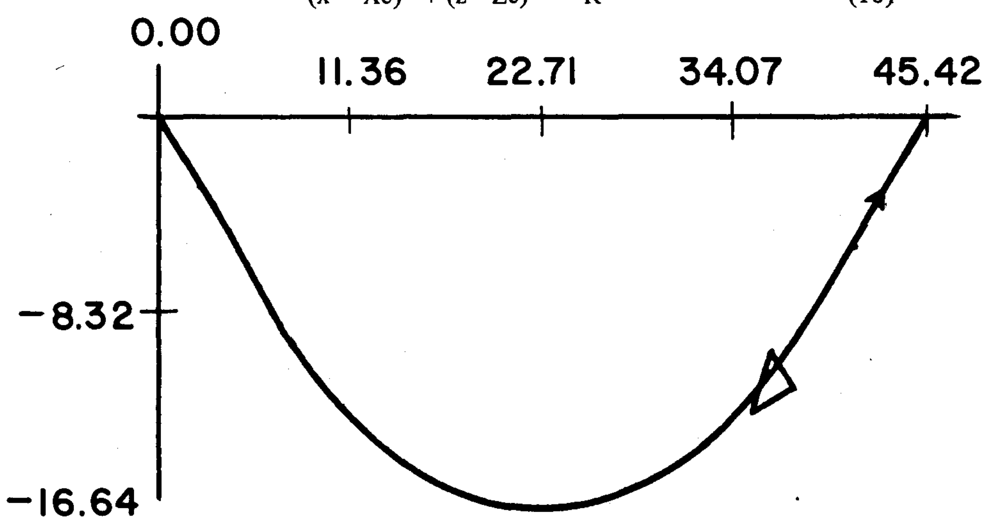

Fig. 1. Infinitesimal triangular region. Inside this region $v(x, z) = vo + b_1x + b_2z$, $b_1, b_2 =$ const.

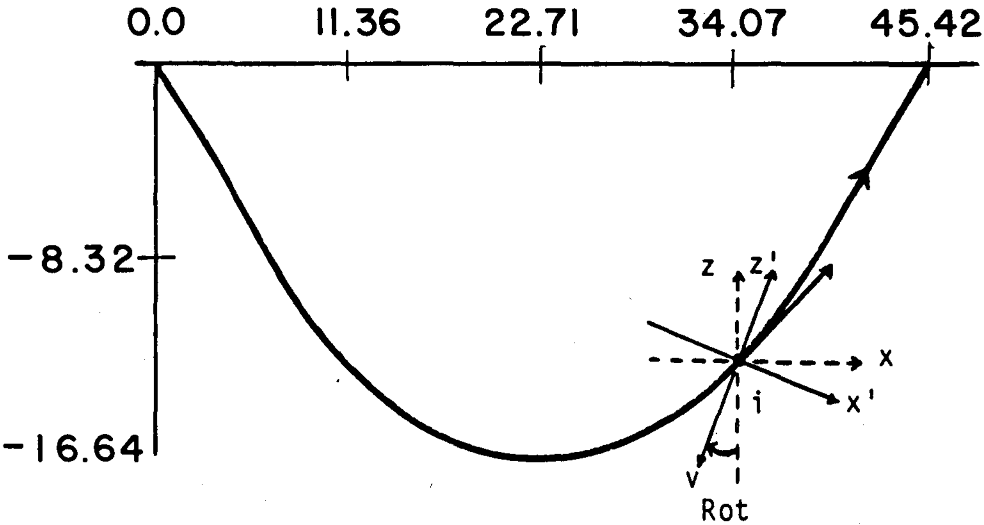

Fig. 2. Local system at point (x, z).

Determination of the displacement

In Marks and Hron (1978) and Chapman and Drummond (1982), the advance of a ray path inside a triangular region is determined by solving for the intercept of the ray with the appropriate side of the triangle. This implies many operations, since there is a system of equations for each side, and various criteria of selection must be used. In the present case, one needs to be concerned only with the physical boundaries, and may drop any other auxiliary boundary. Thus, the local advance of a ray may be determined by means of a specific rule governing the adequacy of its size and direction. The only remaining concern is that of determining if the advance is "good" (final point inside the region) or "bad" (final point outside the region). "Bad" advances are fixed by an interpolation to the adequate boundary, (Fig. 3). This scheme has been previously used (see, for instance Gebrande, 1976).

Let the horizontal displacement be dx. It is convenient to write it as:

$$dx = R/N \tag{11}$$

if it is required that this displacement agrees with the initial direction of the ray (Fig. 2), it is possible to write

$$x_f = x_0 + dx \tag{12}$$

and from (10)

$$z_f = Zc + (R^2 - (x_f - Xc)^2)^{1/2} \tag{13}$$

Once the point (xf, zf) has been correctly determined, the components of slowness are given by (9). As already mentioned, if both, xf and zf are inside the initial region (Fig. 3), the computation of the ray

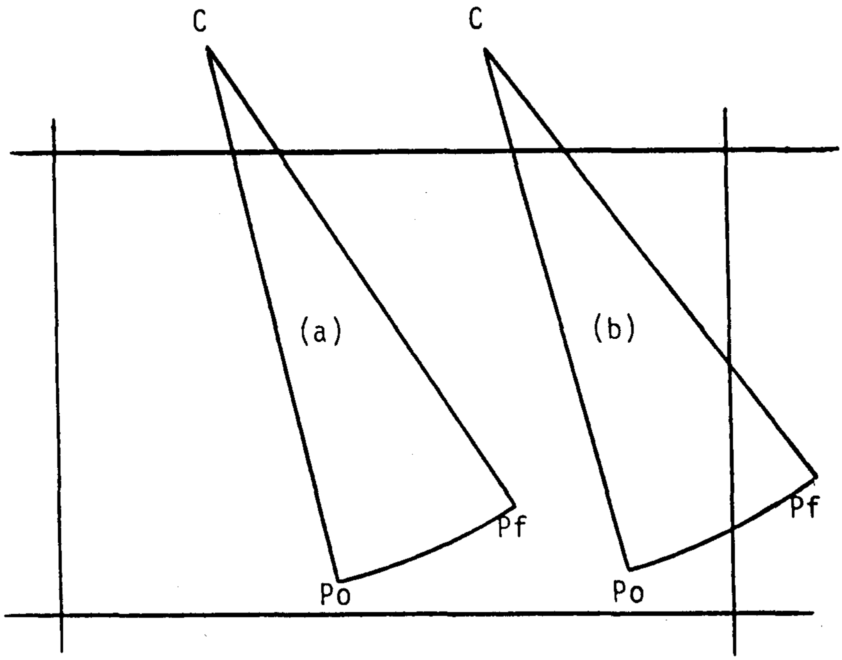

Fig. 3. (a) "Good" displacement. (b) "Bad" displacement.

path may proceed. If either x or z fall outside the initial region, it is necessary to interpolate the ray path to the crossed boundary. The new region is thus determined, and if necessary, Snell's Law is applied at the boundary. The horizontal displacement must satisfy

$$|xf - Xc| \ll R$$

If this condition is not met, it is enough to interchange x and z in (11), (12) and (13) to continue the calculations. This procedure may be seen as the construction of the solution of a differential equation. The proposed method is similar to the implicit Euler's method for solving differential equations of the form

$$\frac{dz}{dx} = f(x, z)$$

but here the solution includes a term $(\frac{d^2 z}{dx^2})$ associated with the curvature. In Euler's method, the increment in x is either constant or interpolated by means of a rule (see, for example, Dalquist and Bjorch, 1974) so that

$$z_f = z_0 + f(x_0, z_0)dx$$

and we get a linear (first order) advance. Other methods achieve higher order approximations using several points (Runge-Kutta methods).

Let

$$dz/dx = f(x, z) \tag{14}$$

be the differential equation of a curve (a ray), from which initial conditions xo, zo, po, qo ($dz/dx = q/p$) are known. The radius of curvature is:

$$R = [1 + f^2(x, z)]^{3/2}/(d^2 z/dx^2) \tag{15}$$

One can say that (15) is an intrinsic property of every real curve of class C(2). The function solving (14) may be interpreted as a curve corresponding to a particular ray in a medium whose velocity is given by (1). Equating (15) and (8) it is possible to obtain

$$d^2 z/dx^2 = (1 + f^2)^{3/2} (-pb_2 + qb_1) \tag{16}$$

This is a second order differential equation for the ray. Its solution exists for cases of geophysical interest, although it may be extremely difficult to obtain for any but the simplest ones. For example, if the medium is homogeneous, (16) reduces to ($b_1 = b_2 = 0$)

$$d^2 z/ dx^2 = 0 \tag{17}$$

whose solution is $z = c_1 x + c_2$ (if the velocity is homogeneous the rays are straight lines). A second case is $b_1 = 0$, $b_2 =$ const., $v = v(z)$. Integration of (17) yields

$$(x - c_1/pb)^2 + (z - c_2)^2 = 1/(pb)^2 \tag{18}$$

agreeing with (10).

A third very simple case is $b_1 =$ const, $b_2 =$ const, which reduces to the case above with $p = p'$. The solution of more complex cases is a subject for future investigation.

Travel time computation

The determination of travel time as the ray goes from (xo, zo) to (xf, zf) is carried out in a simple way in the local system, putting

$$T = \int_0^f dT = \int_0^f R d\theta / v$$

or

$$T = (1/b)[\ln((1 + \sin\theta)/\cos\theta)]_0^f \qquad (19)$$

with

$$\sin\theta' = -\cos i' \qquad \cos\theta' = \sin i'$$

and

$$\theta' = \theta + r$$

where θ is the angle the slowness vector forms with the x-axis and θ' is the angle of the same vector with the local x'-axis (Fig. 2).

As will be seen later, it is not necessary that N, in expression (11), be very large in order to obtain an adequate ray path. With N = 100 is sufficient. For this value of N, $\sin\theta = \theta$ up to the fifth significant figure, which means that, concerning the path, a constant velocity and a constant gradient would give essentially the same result. The travel time, on the contrary, is more sensitive to N at least by an order of magnitude. It is worth noting that the proposed algorithm automatically provides resolution, in the sense that the path is effectively controlled by the combination $1/p'b$, i.e., by the radius of curvature, so that those regions where the radius is small will require more points to specify the path. Caution is recommended in those cases in which the curvature changes sign (Fig. 4), because in the neighbourhood of the inflexion point the radius becomes too large, and the advance must be controlled to prevent divergence in the solution. This is easily achieved by defining a minimum advance, to be used if necessary, instead of that determined by the local radius.

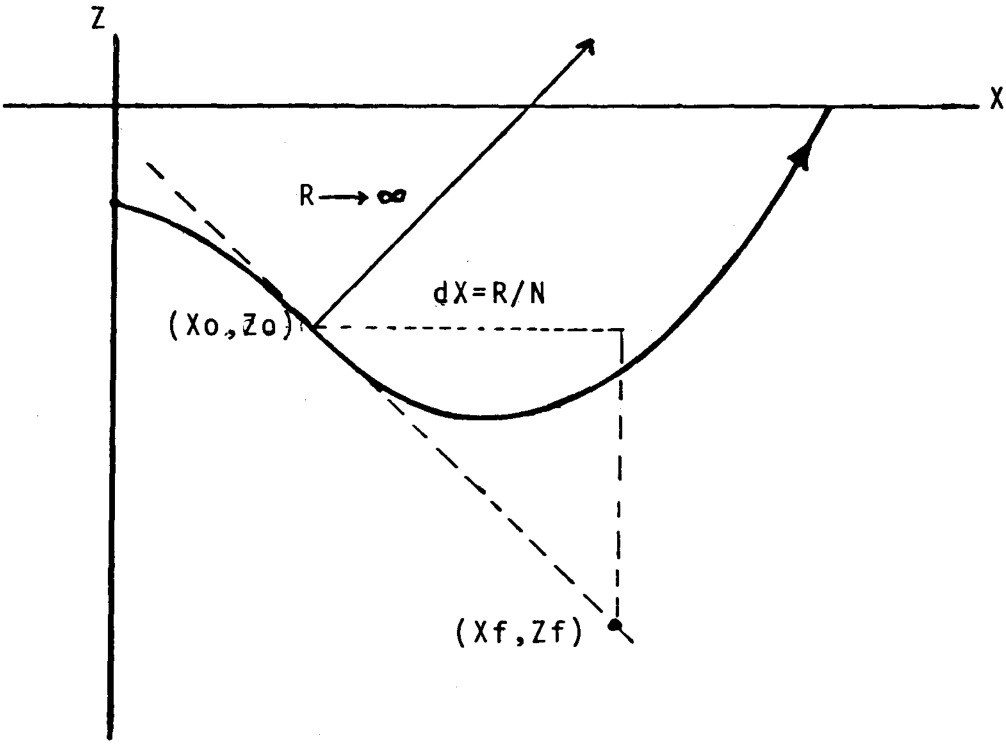

Fig. 4. A ray with a point of inflexion. In a neighbourhood of (xo, zo), R —— ∞.

If the velocity of propagation is a linear function of the coordinates, the solution (12) and (13) is exact, since the Taylor expansion is complete. Later it will be numerically shown that even for non-linear cases the approximation holds good.

3. EXAMPLES

In order to establish comparisons, I have taken the examples contained in Chapman and Drummond (1982), for two reasons: 1) These models illustrate the formation of caustics, and (2) They are very simple to represent analytically.

Model CD1 consists of two layers with no discontinuity in velocity, and the boundary at 10 u. depth. The jump in gradient at this depth is reflected as a caustic at a distance of 45 u. This model is specified by a set of equations and conditions:

$$v = vo = 3.0 \text{ u/sec}$$
$$v = vo + b(z - zo) \tag{20}$$

$$b = -0.25 \text{ u/seg/u}, \; zo = -10u$$

The result is shown in Fig. 5. This case is exact. Fig. 5(a) shows the travel time curve, while Fig. 5(b) illustrates several rays for this model.

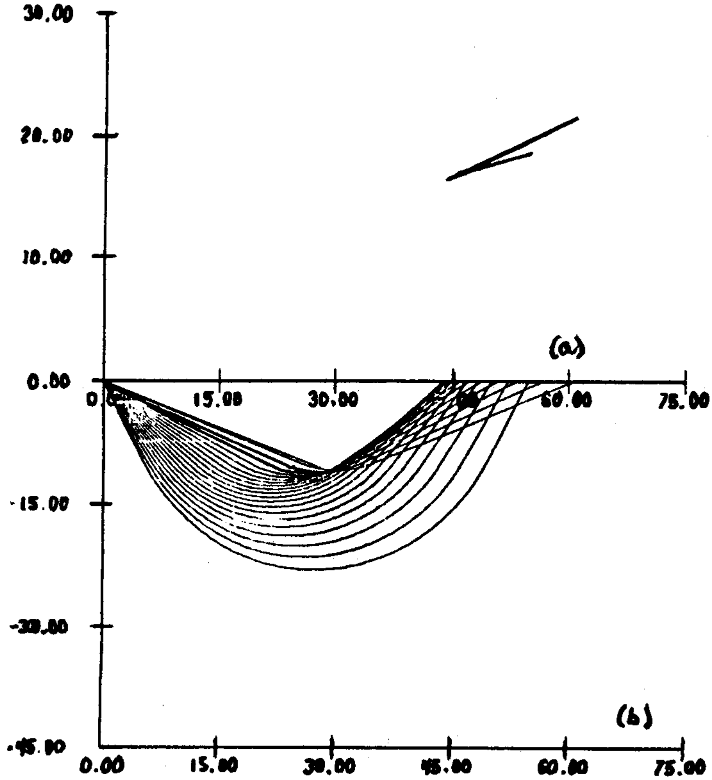

Fig. 5. Model CD1. (a) Raypaths and travel times. (b) Note the caustic close to 45 u.

Model CD2 is similar to the previous one, but in the half space a dome shaped anomaly has been included. In this example, the advan-

The model and computed ray paths are shown in Fig. 6. The features of the ray paths are similar to those reported by Chapman and Drummond, except as regards to numerical details concerned with the specification of the model.

tages of modeling through analytical expressions are clearer. In Chapman and Drummond (1982), the dome has been modeled by modifying the triangles associated with the anomalous region. To change the shape or position of this anomaly, one must reparameterize a considerable portion of the model. In the method I discuss here it is enough to change the value of three parameters: the position of the center of the anomaly (xc), its extent (A) and its amplitude (Δv). The anomaly is modulated horizontally by a hyperbolic secant (Fig. 6) while its posi-

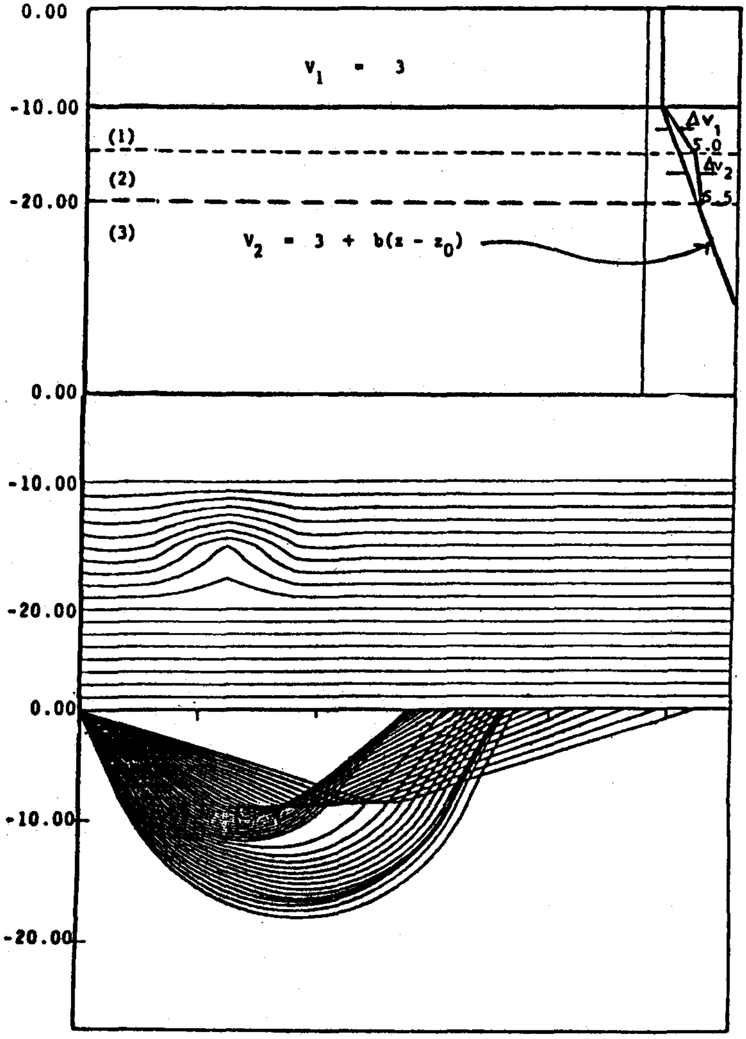

Fig. 6. Model CD2. (a) Velocity as function of depth at x = xc. (b) Isovelocity lines. (c) Ray-paths. The anomaly is efficiently modeled by a hyperbolic secant sech(A(x−xc)).

tion is determined by the pair (xc, za), with $-10u > za > 20u$. The maximum of the anomaly is at $z = -15u$. The chosen analytical formulation of this model is as follows:

$$v = 3.0\ u/s$$

$$v_Z = 3.0 - b(z - zo) \tag{21}$$

$$v = v_Z + \Delta v_i(z)\ \mathrm{sech}\,[A(x - xc)]$$

$$i = 1, 2, 3, \quad \Delta v_3 = 0, \quad xc = 20u.$$

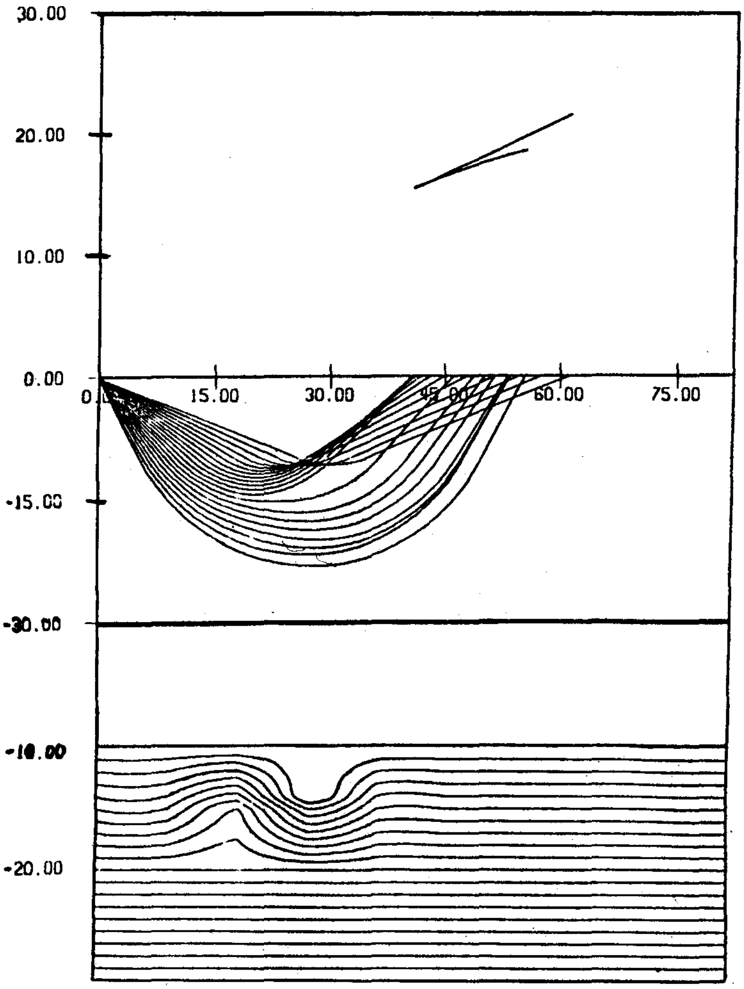

Fig. 7. Model CD3. This model includes a valley. This is well represented by an inverted hyperbolic secant. (a) Travel times for the raypaths in (b). (c) Isovelocity lines.

In model CD3 (Fig. 7), a valley has been included using the same technique (an upside-down dome). An inverted hyperbolic secant function adequately achieves a good representation of the valley. The extent of the bell shape is controlled by the factor A in (21). Examination of results indicates that the method works properly and that differences between these results and those mentioned above are caused by differences in the parameterizing of models (discrete versus continuous case). For further illustration, models CD4 and BV5, the first, consisting of two layers with a discontinuity in the velocity gradient

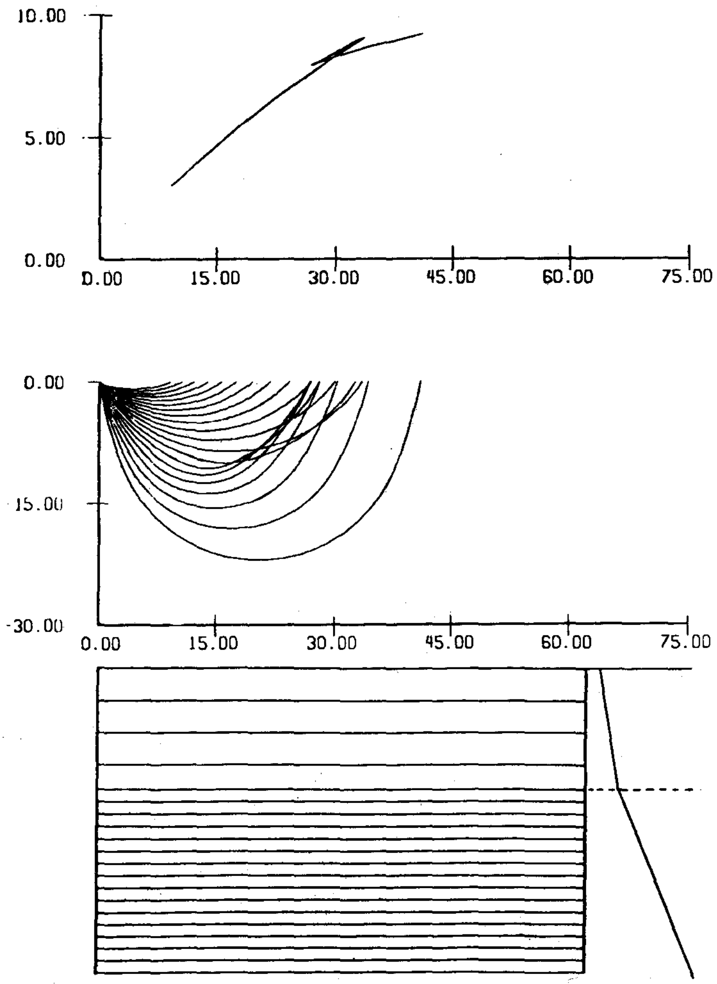

Fig. 8. Model CD4. Two layers with homogeneous gradient, with no discontinuity in velocity The triplication zone is noticeable in the rays, but it is imperceptible in the travel time curve.

(Fig. 8), and the second, of four layers including a low velocity zone between z = – 15 u and z = – 20 u (third layer, Fig. 9) have been included. Figure 10 shows the results obtained varying N from 50 to 10,000 for three different rays in model CD2. From these examples, it is clear that a value of N = 100 is enough to achieve proper results.

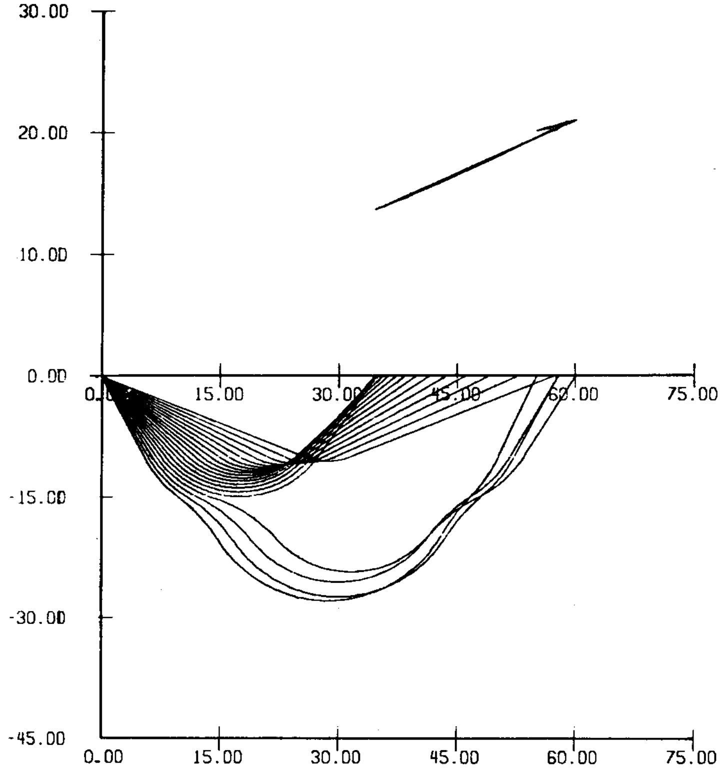

Fig. 9. Model BV5. Four layers including a LVZ between z = -15 u and z = -20 u.

4. THE THREE DIMENSIONAL CASE

This case needs not present special difficulties if one uses the two-dimensional theory developed so far, the fundamental fact being that locally, the ray is an arc of a circle whose radius (of curvature) is $R = 1/p'b'$, where p' is the local horizontal slowness. The local system is built as follows: the ray is contained in an instantaneous plane of propagation, as illustrated in Fig. 11. This plane contains the vectors $\underline{u}$ and grad $\underline{v}$. A unit vector perpendicular to that plane is

$$\tilde{n} = (\underline{u} \times \mathrm{grad}(v))/|\underline{u} \times \mathrm{grad}(v)| = \tilde{j}' \qquad (22)$$

or,

$$\tilde{n} = n1, n2, n3) = \hat{j}$$

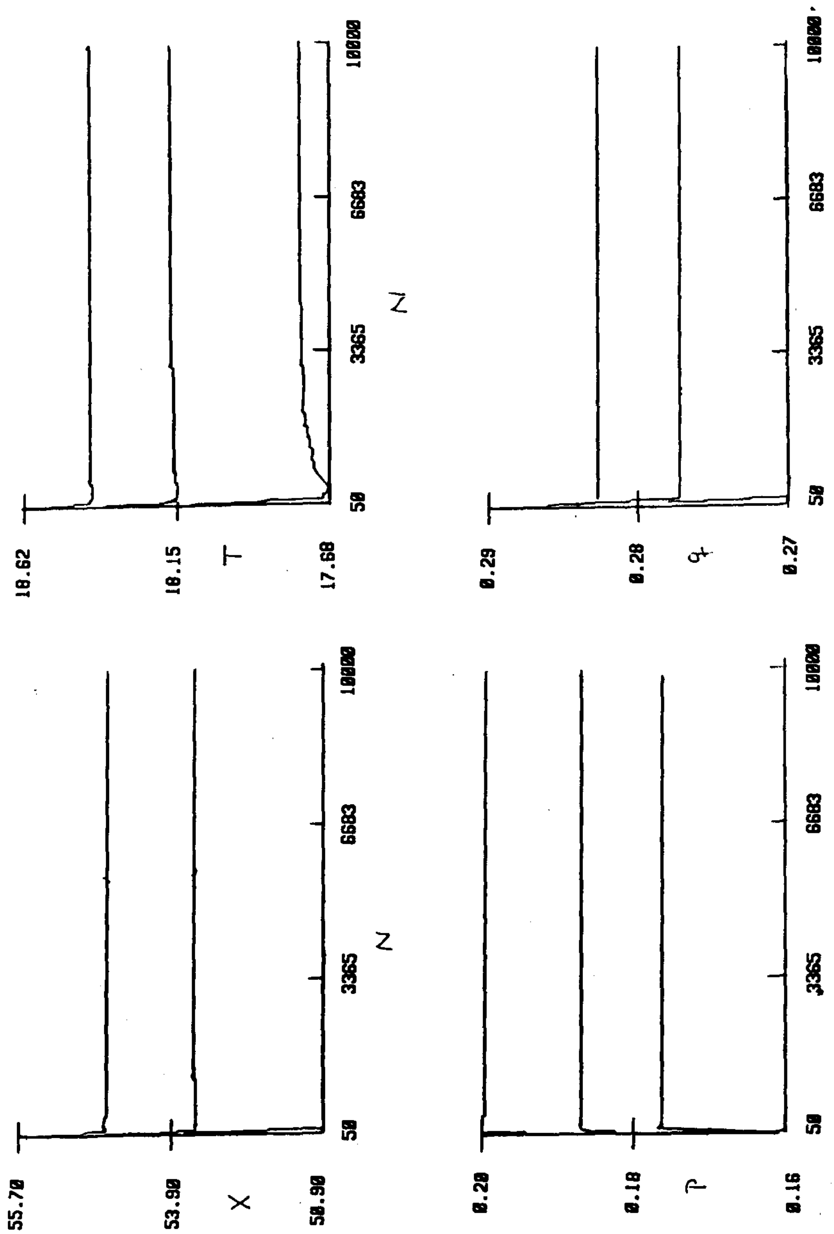

Fig. 10. Convergence of (a) X, (b) T, (c) p, (d) q.

a unit vector in the plane, perpendicular to (22), is

$$\hat{k}' = -\,\mathrm{grad}(v)/|\mathrm{grad}(v)| = (k1, k2, k3) \tag{23}$$

and a unit vector in the plane and normal to (23) and (22) is

$$\hat{i}' = \hat{j}' \times \hat{k}' = (i1, i2, i3) \tag{24}$$

so that $\hat{i}'$, $\hat{j}'$ and $\hat{k}'$ form an orthogonal three-dimensional basis, and the rotation from the external system $(\hat{i}, \hat{j}, \hat{k})$ to the local system $(\hat{i}', \hat{j}', \hat{k}')$ is

$$Rt = \begin{pmatrix} i1 & i2 & i3 \\ n1 & n2 & n3 \\ -b1/b & -b2/b & -b3/b \end{pmatrix} \tag{25}$$

The rotated (local) components of slowness, position or any other vector, may be thus immediately obtained. In particular:

$$p' = pi1 + ri2 + qi3 = \underline{u} \cdot \hat{i}' \tag{26}$$

Also, rotation (25) causes the y-component of every vector contained in the plane of propagation to vanish.

In the local system, the slowness vector has components

$$\underline{u} = (p', 0, q')$$

since in this system velocity is a function only of z′, p′ is conserved, and the path of the ray is an arc of a circle. At this point, the three-dimensional problem has been reduced to a two-dimensional problem, already solved in the last section. One can then generate the "displacement" of the ray as described, and obtain the final point:

$$\underline{r}f = (x'f, D, z'f) \tag{27}$$

and with the values of x′f, z′f and X′c and Z′c (the local center of the circle), obtain the value q′f. The external corresponding quantities are then obtained through the inverse rotation.

$$\begin{aligned} \underline{r}f &= (xf, yf, zf) = Rt^{-1}(\underline{r}f) \\ \underline{u}f &= (pf, rf, qf) = Rt^{-1}(\underline{u}'f) \end{aligned} \tag{28}$$

where

$$\underline{u}f = p', 0, q'f), p' = \text{const}$$

The set of quantities (xf, yf, zf, pf, rf, qf) specifying the new point is now complete.

Iteration of this procedure, exchanging the final point with the initial one, will determine the ray path and the travel time in an analogous fashion to the two-dimensional case. This is possible because during propagation over the instantaneous plane, a ray tube experiences no expansion or contraction in a direction perpendicular to the plane. This implies a succession of directions in which there is no distortion. In the ensuing paragraphs, simple examples illustrating what is to be expected

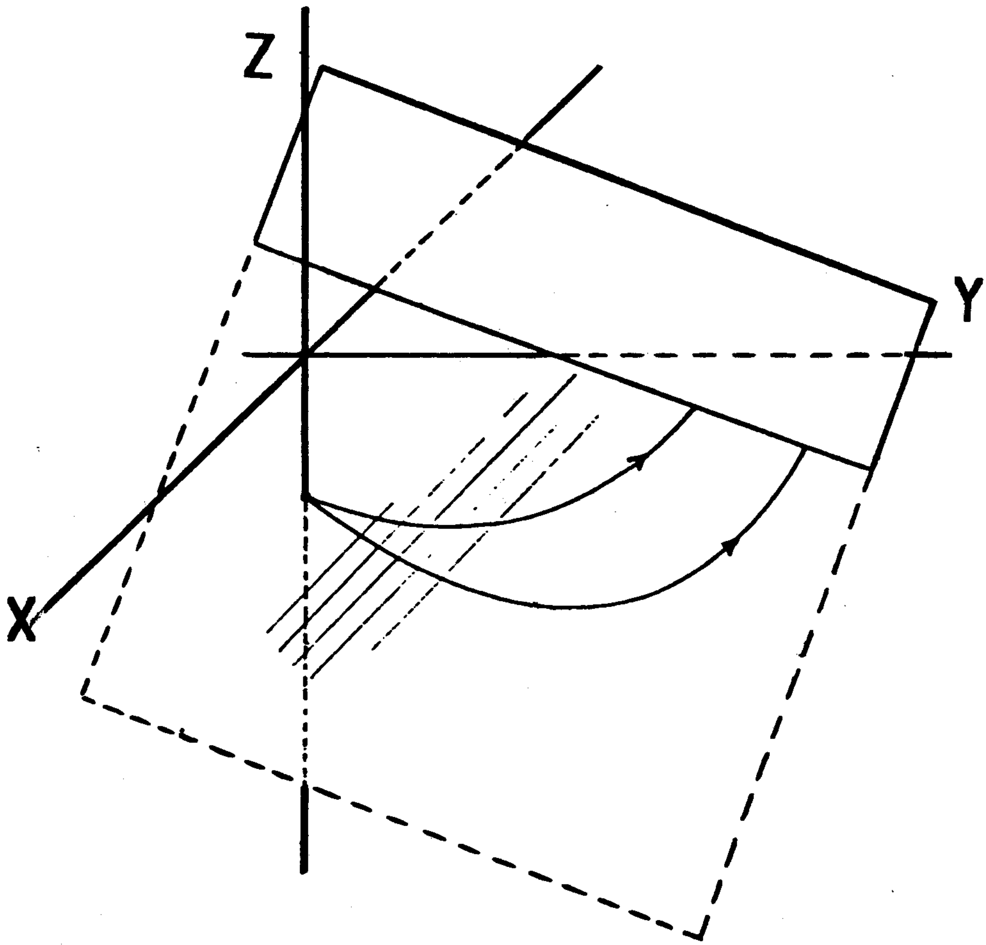

Fig. 11. Instantaneous plane of propagation and local system. The vector u and grad(v) are contained in the plane.

from this method is explained. A program to draw the ray paths and the equal travel time curves (surface isochrones) is being prepared.

5. CONCLUSIONS

The present method is an extension of Euler's implicit method for numerically solving a certain type of differential equations of the first order. In this development, use has been made of the fact that the first and second derivatives of the path coincide with the corresponding derivatives of the oscullating circle, so that curvature is automatically taken into account, and consequently, the error is less than in other methods. From this, it is possible to show that the error is $\sum_{i=1}^{n} R_i/N^3$ where n is the number of points of a path. Since, in general, $n \sim N$, we conclude that the error behaves as $\langle R \rangle / N^2$ with $\langle R \rangle$ the average radius of the ray path. The procedure used in constructing the "displacement" provides the method with an intrinsic resolving mechanism, in the sense that paths with greater curvature will automatically have more points.

The computation of travel time for the three-dimensional case reduces to that of two-dimensions at each point of the path. The computational algorithm is fast and accurate. The simplicity of the formulation given here allows the use of analytical expressions to provide smooth data to use in further calculations, as for example, the construction of synthetic seismograms, and continuous inversion schemes. The author feels that the disadvantages of an analytical representation for realistic velocity structures are a matter of research, and that, in the long run, this mode of operation will prove to be more convenient than the more general discrete approaches.

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