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CALCULATION OF GRAVITY EFFECTS OF TRIDIMENSIONAL STRUCTURES BY ANALYTICAL INTEGRATION OF A POLYHEDRIC APPROXIMATION AND APPLICATION TO THE INVERSE PROBLEM

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RESUMEN

Se deducen expresiones analíticas de rápido procesamiento en máquina, para el cálculo de la anomalía gravimétrica producida en un punto exterior por un modelo de cuerpo tridimensional homogéneo, definido al dar varias secciones verticales paralelas de contorno poligonal vinculadas lateralmente por caras triangulares.

Se emplea luego este tipo de cuerpo en el modelado de soluciones del problema gravimétrico inverso, efectuándose algunas sugerencias respecto de la convergencia.

Se presentan también ejemplos de cálculos directos e inversos.

ABSTRACT

Analytical expressions, rapid in computer time, are derived to calculate the gravity anomaly on an external point due to a model of homogeneous tridimensional body defined by providing several parallel vertical polygonal contour sections laterally assembled by triangular faces.

Such type of body is then used for modeling solutions of the inverse gravity problem and some suggestions on convergence are made.

Examples of direct and inverse calculations are also presented.

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INTRODUCTION

The general case of analytical calculation of gravity effects caused by homogeneous tridimensional structures of arbitrary shape – 3-D direct problem – has been studied by Barnett (1976) and Okabe (1979) among others. The first author deduces expressions for polyhedric bodies with triangular faces. The second author develops expressions for polyhedric bodies with faces having an arbitrary number of sides.

The case of prismatic bodies with constant polygonal cross section and finite strike length – $2\frac{1}{2}$-D direct problem – has been studied, among other authors by Rasmussen and Pedersen (1979) and Caddy (1980).

In a previous work – hereinafter called "Paper I" – Guspí, Introcaso and Huerta (1984) developed formulae to calculate the vertical gravity anomaly due to a vertical

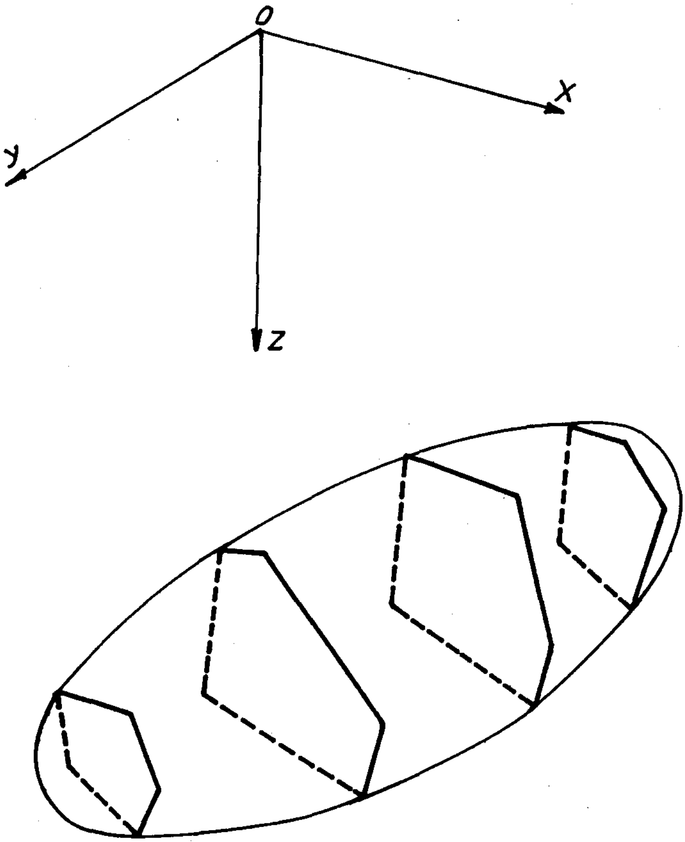

Fig. 1. Discretization of a tridimensional body by polygonal laminae.

polygonal lamina. In that work, a tridimensional body is sampled by taking a certain number of parallel vertical polygonal sections or laminae (Fig. 1) and the effect of each one is calculated separately. Then a numerical integration is performed to obtain the attraction of the whole body.

Such a discretization led the authors to find out a method for analytically integrating the effect of those laminae, and a solution was found under the hypothesis that the lateral surface of the body is constituted by triangular faces. No rotation of coordinates is performed to solve the integration.

STATEMENT OF PROBLEM

As in Figure 1 we approximate the body by giving several vertical polygonal sections, parallel to the XZ plane, and we consider that each of them has the same number n of sides. (This last constraint was not necessary in Paper I's method).

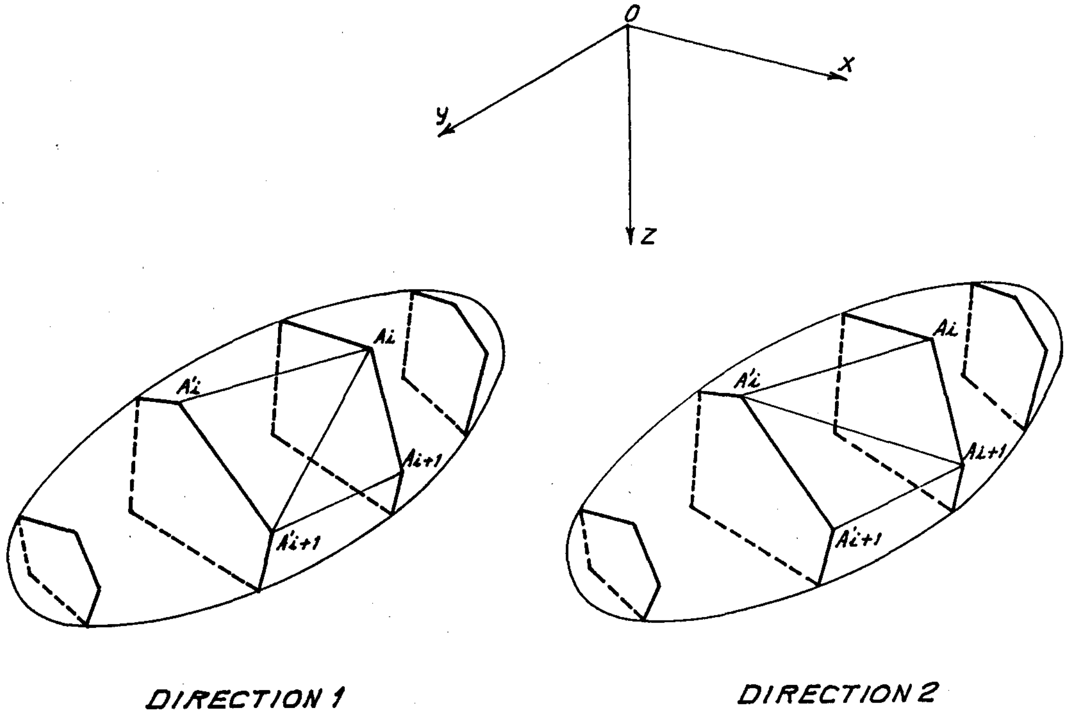

Fig. 2. Triangulation of lateral surface can be made in two directions.

The lateral surface of the body can be approximated by considering that a pair of consecutive vertices on one section and its homologue pair on a consecutive section, define two triangular plane faces. Notation and position of coordinate axes are defined in Figure 2. If we unite each vertex A_i of the body with the vertex A'_{i+1} we have the direction 1 of triangulation. If we unite A_{i+1} with A'_i we have the direction 2. Of course, the result obtained will vary with the direction choosen.

Since the total anomaly caused by the body is the sum of the anomalies produced by the blocks bounded by two consecutive sections, we will study the effect of a single block.

Let K be a block defined by the $y = y_1$ and $y = y_2$ sections (Fig. 3). The vertical anomaly caused by K at the origin of coordinates is

$$\Delta g_K = G\rho \int\int\int_K \cdot \frac{z\, dx\, dy\, dz}{(x^2 + y^2 + z^2)^{3/2}} = G\rho \int_{y_1}^{y_2} dy \int\int_{\sigma(y)} \frac{z\, dx\, dz}{(x^2 + y^2 + z^2)^{3/2}} \qquad (1)$$

where G is the Newton's gravitational constant, ρ is the density of the body and $\sigma(y)$ is an intermediate section of the block produced by a vertical plane parallel to XZ whose y coordinate varies between y_1 and y_2.

As in Paper I we define

$$V(y) = \int\int_{\sigma(y)} \frac{z\, dx\, dz}{(x^2 + y^2 + z^2)^{3/2}} \qquad (2)$$

– contribution from $\sigma(y)$ lamina – and then

$$\Delta g_K = G\rho \int_{y_1}^{y_2} V(y)\, dy. \qquad (3)$$

If we look at Figure 3 we can see that $\sigma(y)$ is a polygon of 2n sides, and we can calculate its contribution V(y) by summing the individual contribution of its sides, as made in Paper I; that is,

$$V(y) = \sum_{i=1}^{2n} V_i(y) \qquad (V_i(y)\text{: contribution of i-th side}). \qquad (4)$$

Then in (3)

$$\Delta g_K = G\rho \int_{y_1}^{y_2} \sum_{i=1}^{2n} V_i(y)\, dy = G\rho \sum_{i=1}^{2n} \int_{y_1}^{y_2} V_i(y)\, dy. \tag{5}$$

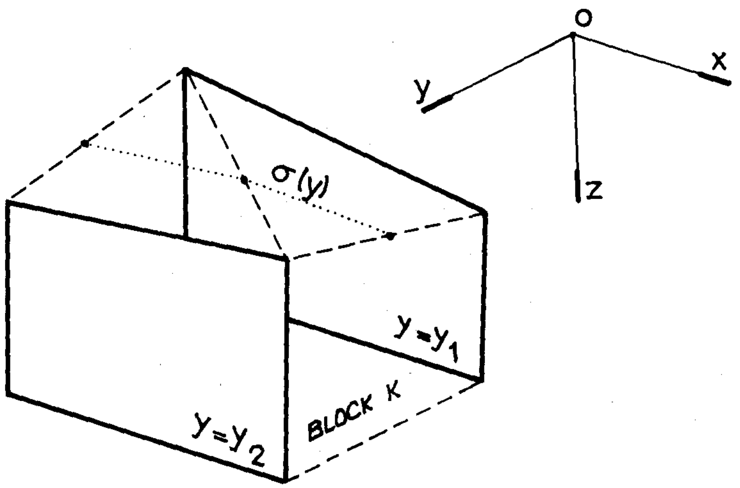

Fig. 3. Block and $\sigma(y)$ section.

When y varies from y_1 to y_2 each side of the polygonal section $\sigma(y)$ runs over a triangular face of the block. Then

$$\tau_i = \int_{y_1}^{y_2} V_i(y)\, dy \tag{6}$$

represents the contribution of the triangle face containing the i-th side of the vertical intermediate sections.

CALCULATION OF THE ANOMALY OF A TRIANGULAR FACE

Let us consider a triangle (Fig. 4) with the following vertices:

$$A\,(x_a, y_1, z_a)$$
$$B\,(x_b, y_2, z_b)$$
$$C\,(x_c, y_2, zc)$$

which represents a face of the polyhedron. The $\overline{BC}$ side is normal to the Y axis.

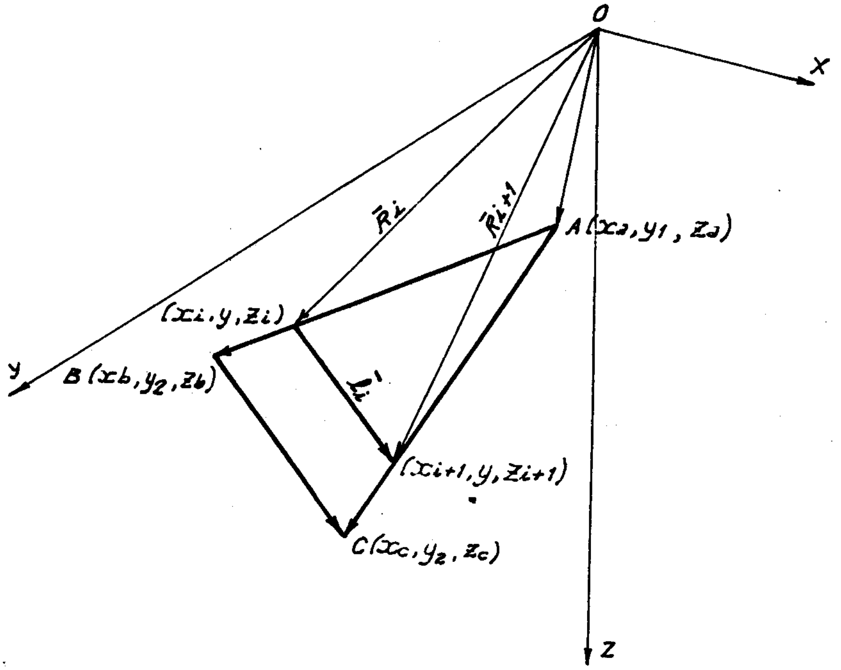

Fig. 4. Calculation of the anomaly of a triangular face.

The side of a $\sigma(y)$ lamina belonging to this face ($\bar{l}_i$ vector) is then parallel to the $\overline{BC}$ side and its extremes have coordinates

$$(x_i, y, z_i) \qquad (x_{i+1}, y, z_{i+1})$$

To calculate $V_i(y)$ from (6) we need to apply the expression developed in Paper I:

$$V_i(y) = \frac{\Delta x_i}{l_i} \ln \frac{R_{i+1} + l_i + \frac{T_i}{l_i}}{R_i + \frac{T_i}{l_i}}$$

being

$$\Delta x_i = x_{i+1} - x_i$$
$$\Delta z_i = z_{i+1} - z_i$$

$$l_i = \sqrt{\Delta x_i^2 + \Delta z_i^2} = \left|\bar{l}_i\right|$$

$$T_i = x_i \Delta x_i + z_i \Delta z_i = \bar{R}_i \cdot \bar{l}_i \qquad (7)$$

(• : scalar product)

$$R_i = \left|\bar{R}_i\right| = \sqrt{x_i^2 + y^2 + z_i^2}$$

$$R_{i+1} = \left|\bar{R}_{i+1}\right| = \sqrt{x_{i+1}^2 + y^2 + z_{i+1}^2}$$

In order to perform integration with respect to y, we introduce a parameter t which ranges from 0 to 1.

So,

$$R_{i+1} = \left|\overline{OA} + t\,\overline{AC}\right| = \sqrt{\left|\overline{AC}\right|^2 t^2 + 2\,(\overline{OA}\cdot\overline{AC})\,t + \left|\overline{OA}\right|^2}$$

$$R_i = \left|\overline{OA} + t\,\overline{AB}\right| = \sqrt{\left|\overline{AB}\right|^2 t^2 + 2\,(\overline{OA}\cdot\overline{AB})\,t + \left|\overline{OA}\right|^2}$$

$$l_i = t\,\left|\overline{BC}\right|$$

$$T_i = \bar{R}_i \cdot t\,\overline{BC} = (\overline{OA} + t\,\overline{AB})\cdot t\overline{BC} \qquad (8)$$

$$\frac{T_i}{l_i} = \frac{\overline{AB}\cdot\overline{BC}}{\left|\overline{BC}\right|}\,t + \frac{\overline{OA}\cdot\overline{BC}}{\left|\overline{BC}\right|}$$

$$l_i + \frac{T_i}{l_i} = \frac{\overline{BC}\cdot\overline{BC} + \overline{AB}\cdot\overline{BC}}{\left|\overline{BC}\right|}\,t + \frac{\overline{OA}\cdot\overline{BC}}{\left|\overline{BC}\right|} = \frac{\overline{AC}\cdot\overline{BC}}{\left|\overline{BC}\right|}\,t + \frac{\overline{OA}\cdot\overline{BC}}{\left|\overline{BC}\right|}$$

$$\Delta x_i = (x_c - x_b)\,t$$

$$\frac{\Delta x_i}{l_i} = \frac{x_c - x_b}{\left|\overline{BC}\right|}$$

$$\frac{dy}{dt} = y_2 - y_1$$

Then by replacing expressions (8) into (7) and by integrating with respect to t we obtain

$$\tau_i = (y_2 - y_1)\frac{x_c - x_b}{|\overline{BC}|}\int_0^1 \ln \frac{\frac{\overline{AC}.\overline{BC}}{|\overline{BC}|}t + \frac{\overline{OA}.\overline{BC}}{|\overline{BC}|} + \sqrt{|\overline{AC}|^2 t^2 + 2(\overline{OA}.\overline{AC})t + |\overline{OA}|^2}}{\frac{\overline{AB}.\overline{BC}}{|\overline{BC}|}t + \frac{\overline{OA}.\overline{BC}}{|\overline{BC}|} + \sqrt{|\overline{AB}|^2 t^2 + 2(\overline{OA}.\overline{AB})t + |\overline{OA}|^2}}\, dt =$$

$$= (y_2 - y_1)\frac{x_c - x_b}{|\overline{BC}|}\left[\int_0^1 \ln\left(\frac{\overline{AC}.\overline{BC}}{|\overline{BC}|}t + \frac{\overline{OA}.\overline{BC}}{|\overline{BC}|} + \sqrt{|\overline{AC}|^2 t^2 + 2(\overline{OA}.\overline{AC})t + |\overline{OA}|^2}\right)dt -\right.$$

$$\left. - \int_0^1 \ln\left(\frac{\overline{AB}.\overline{BC}}{|\overline{BC}|}t + \frac{\overline{OA}.\overline{BC}}{|\overline{BC}|} + \sqrt{|\overline{AB}|^2 t^2 + 2(\overline{OA}.\overline{AB})t + |\overline{OA}|^2}\right)dt\right] \qquad (9)$$

That is:

The contribution of the face can be expressed by the difference between two integrals having a similar shape, the first concerning the $\overline{AC}$ side and the second the $\overline{AB}$ side.

We can summarize both integrals as

$$I = \int_0^1 \ln(\alpha t + \beta + \sqrt{at^2 + bt + d})\, dt. \qquad (10)$$

Details of analytical integration are given in Appendix, and the following result is obtained:

$$I = \frac{1}{\sqrt{a}}\left[\left(v_2 - \frac{pq}{1-p^2}\right)\ln\left(pv_2 + q + \sqrt{v_2^2 + c}\right) - \right.$$

$$- \left(v_1 - \frac{pq}{1-p^2}\right)\ln\left(pv_1 + q + \sqrt{v_1^2 + c}\right) - v_2 + v_1 +$$

$$+ \frac{q}{1-p^2}\ln\frac{v_2 + \sqrt{v_2^2 + c}}{v_1 + \sqrt{v_1^2 + c}} +$$

$$\left. + \frac{2w}{1-p^2}\left(\tan^{-1}\frac{(1+p)(v_2 + \sqrt{v_2^2+c}) + q}{w} - \tan^{-1}\frac{(1+p)(v_1 + \sqrt{v_1^2+c}) + q}{w}\right)\right] \qquad (11)$$

with $v_1 = \frac{b}{2\sqrt{a}}$ $\quad v_2 = \sqrt{a} + \frac{b}{2\sqrt{a}}$

$p = \frac{\alpha}{\sqrt{a}}$ $\quad q = \beta - \frac{\alpha b}{2a}$

$c = d - \frac{b^2}{4a}$ $\quad w = \sqrt{c(1-p^2) - q^2}$.

Further simplification can be introduced by making the difference between integrals.

The term $\frac{1}{\sqrt{a}}(-v_2+v_1) = \frac{1}{\sqrt{a}}(-\sqrt{a} - \frac{b}{2\sqrt{a}} + \frac{b}{2\sqrt{a}}) = -1,$ is cancelled.

The term $\frac{1}{\sqrt{a}}\, v_1 - \frac{pq}{1-p^2})\ \ln(pv_1 + q + \sqrt{v_1^2 + c})$ (12)

can be transformed into $\frac{b - 2\alpha\beta}{2(a - \alpha^2)}\ \ln(\beta + \sqrt{d})$;

but β and d are the same on both integrals. Otherwise

$$b - 2\alpha\beta \text{ (on the first integral)} = 2(\overline{OA}.\overline{AC}) - 2\,\frac{\overline{AC}.\overline{BC}}{|\overline{BC}|}\,\frac{\overline{OA}.\overline{BC}}{|\overline{BC}|}$$

and by making $\overline{AC} = \overline{AB} + \overline{BC}$ the result obtained is equal to $b - 2\alpha\beta$ (on the second integral).

In a similar manner
$2(a - \alpha^2)$ (on the first integral) = $2(a - \alpha^2)$ (on the second integral).

Hence, the whole term (12) is removed.

Also, since factor $\frac{1}{\sqrt{a}}(v^2 - \frac{pq}{1-p^2})$
is equal on both integrals, the difference of logarithms in the first terms can be computed as the logarithm of a quotient.

COMPUTER TIME

For a given body, formula (9), where integrals are evaluated by using (11) and its simplifications, must be applied to every triangle of the lateral surface.

We have considered a set of several bodies having different shape – quantity of sections, number and position of vertices, etc. – and we have calculated on the computer, gravity anomalies at different stations, by employing the here proposed method.

Then, the same calculations were made by applying to the bodies the more general methods given by Barnett (1976) and Okabe (1979).

Numerical results were always coincident, but the computer time spent by our method was about 30% less than the time employed by the other two methods which was found to be approximately equivalent in computer time for this type of models.

NUMERICAL EXAMPLE 1

Let us consider an homogeneous block having a density contrast of 1 g/cm^3 whose shape is described in Figure 5.

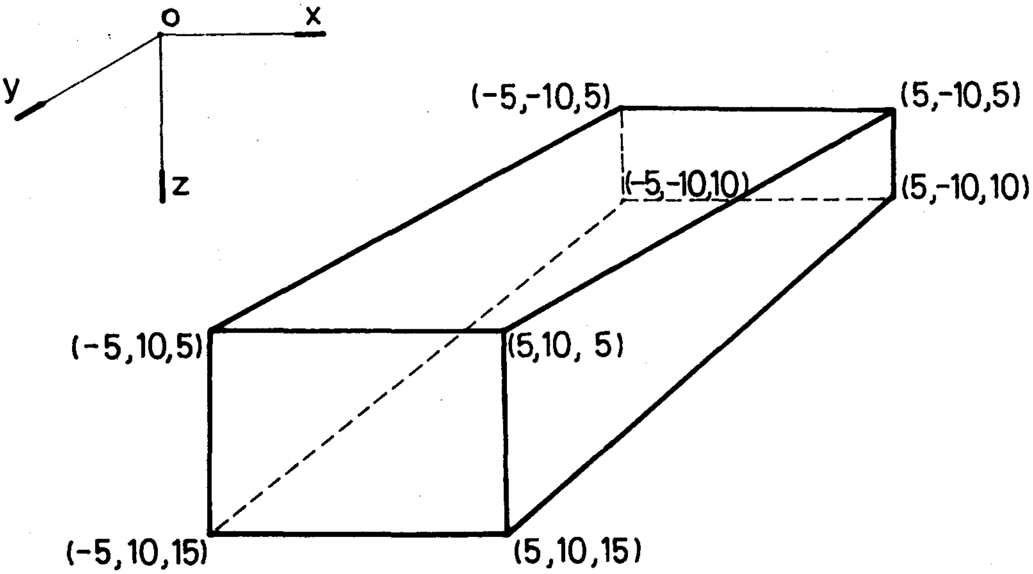

Fig. 5. Numerical example 1. Vertex coordinates are in km.

We calculate vertical anomalies due to that block at the origin of coordinates and at the four points of the XY plane located on the vertical of the vertices.

The limits of integration are the two faces parallel to the XZ plane and we triangulate the remaining faces (lateral surface). Since these faces are all plane, triangles fit exactly the body's shape and the result is the same for both directions of triangulation.

Computed values are shown in Table 1.

Table 1

Computed values for numerical example 1

Point			Computed anomaly
x	y	z	(mGal)
0	0	0	81.04
−5	−10	0	35.70
5	−10	0	35.70
−5	10	0	41.06
5	10	0	41.06

NUMERICAL EXAMPLE 2

Example 1's block was divided into two irregular bodies whose vertices are defined along five vertical sections parallel to the XZ plane, as shown in Figure 6. We compute the anomalies, separately for each body, at the same points as in Example 1.

In this case, different directions of triangulation lead to different results for each body, and this is pointed out in Table 2.

The lower surface of the upper body agrees exactly with the upper surface of the lower body when triangulation is made in different directions on each body. Then, cross sums of contributions, as made in the table, equal the effect of the whole body which, as said, does not depend upon triangulation.

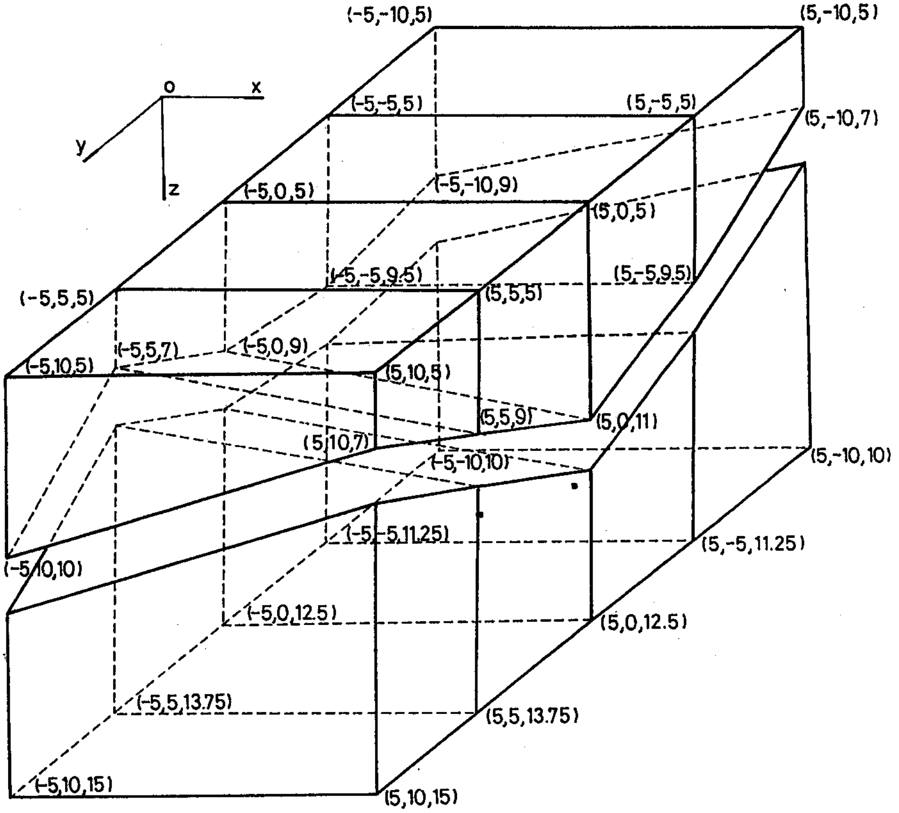

Fig. 6. Numerical example 2. Decomposition of example 1's block.

Table 2

Computed values for numerical example 2.

Point			Triangulation Direction 1		Triangulation Direction 2		Whole body = (a) + (d) = (b) + (c)
x	y	z	(a) Upper body	(b) Lower body	(c) Upper body	(d) Lower body	
0	0	0	54.56	26.49	54.55	26.48	81.04
–5	–10	0	23.37	11.30	24.40	12.33	35.70
5	–10	0	23.00	11.64	24.06	12.70	35.70
–5	10	0	22.89	20.08	20.98	18.17	41.06
5	10	0	22.99	20.07	20.99	18.07	41.06

APPLICATION TO THE INVERSE PROBLEM

A model of homogeneous body with known density contrast having a fixed top surface and a variable bottom surface defined by the z coordinate of a set of points whose x and y coordinates are fixed, is common in gravity interpretation. (Vice-versa, the bottom surface can be fixed and the top surface, variable).

The type of body presented in this paper is very suitable for such a model.

In fact, as in Figure 7, one can construct a series of polygonal vertical sections adapted to the desired model and determine which vertices are to be fixed and which are to be variable (i.e. upper vertices fixed, lower vertices variable).

As it is known, ambiguity in gravity interpretation does not allow – theoretically or practically – all the vertices to be variable.

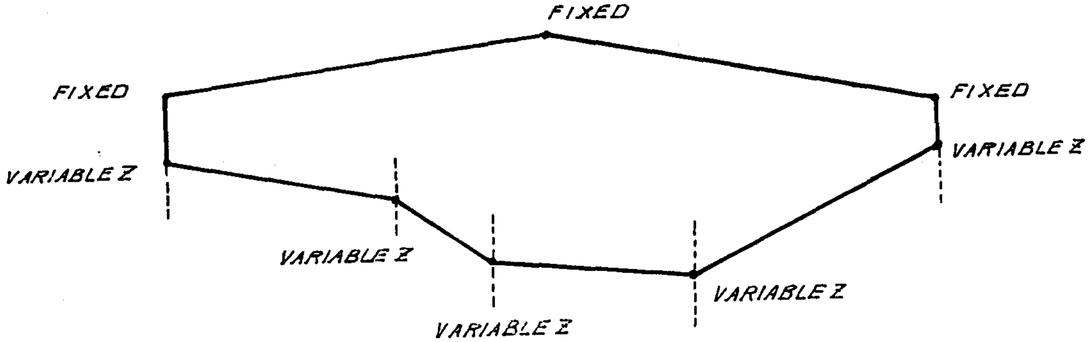

Fig. 7. One of the vertical sections of the model.

Suppose that we have measured gravity at n stations, obtaining a vector of anomalies

$$\bar{g} = \begin{pmatrix} g_1 \\ g_2 \\ \cdot \\ \cdot \\ g_n \end{pmatrix}$$

Suppose also that our model depends upon m parameters, i.e. the z coordinate of m variable vertices, which define a vector

$$\bar{z} = \begin{pmatrix} z_1 \\ z_2 \\ . \\ . \\ z_m \end{pmatrix}$$

We can compute the anomalies caused by the model by applying the expressions developed in previous paragraphs, and the result is a vector of computed anomalies

$$\bar{g}^* (\bar{z}) = \begin{pmatrix} g_1^* \\ g_2^* \\ . \\ . \\ g_n^* \end{pmatrix}$$

We search for a model minimizing the objective function

$$S(\bar{z}) = \sum_{i=1}^{n} [g_i - g_i^*(\bar{z})]^2$$

that is, we need to solve a non-linear least-squares problem.

One of the ways of solving it is by the Gauss - Newton method combined with the Marquardt's (1963) algorithm.

Starting from a $\bar{z}^{(o)}$ guessed solution, one constructs at each iteration a normal m x m system of linear equations in the $\delta\bar{z}$ increments to be added to the former approximation for obtaining a new one

$$J^t J \delta \bar{z} = J^t \bar{g} \tag{13}$$

where J is the Jacobian matrix having at the i-th row, j-th column, the partial derivative

$$\frac{\partial g_i^*}{\partial z_j} .$$

In practice it is convenient to compute numerically the derivatives, by recalculating (11) with a small increment on the z coordinate of each variable vertex of the triangle, since an analytical derivative expression derived from (11) should be very much longer in computer time than (11) itself.

Marquardt's algorithm adds a positive constant λ to all diagonal terms of $J^t J$ in (13). The initial value of λ is arbitrary, and the normal system remains

$$(J^t J + \lambda I)\delta \bar{z} = J^t \bar{g} \qquad (14) \quad \text{(I: identity matrix)}$$

Independent terms of (14) are proportional to ∇S (gradient of the objective function), and their quadratic mean, Q, provides a measure of convergence. In fact, if Q = 0 (in practice less than a tolerance), the gradient is zero and a minimum is reached.

At each iterative step if S (or Q) decreases, λ must be made smaller, i.e., by dividing it by 10. The process tends to the Gauss - Newton method. But if S (or Q) increases, the criterion is to augment λ, i.e. by multiplying it by 10, and then go back to the former step. The process tends to the steepest-descent method.

A flowchart describing the solution of the inverse problem by applying this algorithm is presented in Figure 8.

The above described procedure is found very useful in avoiding divergence when the initial model is far from the solution, but convergence is poor when the matrix of the normal system is ill-conditioned. An ill-conditioned matrix can be obtained not only when $m > n$, but also in most cases of close proximity of the vertices defining the unknown surface.

For this case other procedures can be useful:

a) Generalized matrix inversion. See Pedersen (1977), Enmark (1981), Lines and Treitel (1984) among others.

b) As proposed by Guspí (1984) for the 2-D inverse problem, to add linear constraints in order to limit the free movement or oscillation of the points. These constraints may establish that the vertices of the triangles defining the unknown part of the body must lie over a surface defined by p parameters, being $p \ll m$. The surface can be a polynomial surface of given degree, or a linear combination of sampling functions given, etc.

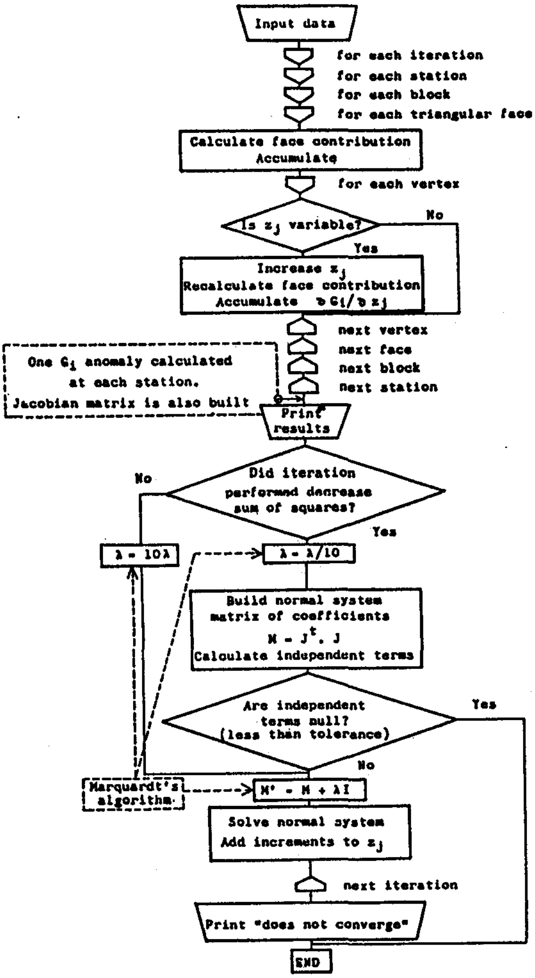

Fig. 8. Synthesis of flowchart followed to solve the inverse problem.

NUMERICAL EXAMPLE 3

We consider as measured anomalies those five exact values calculated in Example 1.

Table 3
Computed values for numerical example 3

Iteration No.	z(−5, −10)	z(5, −10)	z(−5, 10)	z(5, 10)	Gravity anomalies in mGal at stations				
					(0, 0, 0)	(−5, −10, 0)	(5, −10, 0)	(−5, 10, 0)	(5, 10, 0)
1	7.00	7.00	7.00	7.00	31.16	12.87	12.87	12.87	12.87
2	11.49	9.32	11.33	13.50	71.04	31.98	31.30	34.04	34.50
3	10.69	9.92	14.08	15.53	80.15	35.62	35.47	40.28	40.51
4	10.01	10.00	14.95	15.06	81.01	35.69	35.69	41.03	41.05
5	10.00	10.00	15.00	15.00	81.04	35.70	35.70	41.06	41.06

We suppose that the body causing the anomalies is the block in Example 1, but suppose that we only know density and upper surface (coordinates of the four upper vertices).

Then, in order to determine the lower surface (coordinates of the four lower vertices) we apply the Gauss - Newton method with Marquardt's algorithm starting from an initial approach z_{lower} = constant = 7 km.

Table 3 shows convergence through 5 iterations. Since initial and final models are symmetric, the non-symmetry observed on intermediate approximations is due to the influence of triangulation direction on partial derivatives.

CONCLUSIONS

Exact expressions for computing gravity effects of finite structures approximated by a model of polyhedric body have been developed. No rotation of coordinates is performed and formulae are rapid in computer time. The type of body proposed is very suitable for modeling solutions in most cases of inverse problems, and unknown vertices (usually the lower or upper surface of the body) may have different proximities. The Gauss- Newton method with Marquardt's algorithm provides good convergence even from relatively distant input if vertices are not too close. Some concepts have been mentioned concerning the close vertices case.

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APPENDIX

ANALYTICAL INTEGRATION

In order to compute

$$I = \int_0^1 \ln(\alpha t + \beta + \sqrt{at^2 + bt + d})\, dt$$

with $a > 0$ and $b^2 - 4ad < 0$

we firstly substitute

$$t = \frac{1}{\sqrt{a}}\left(v - \frac{b}{2\sqrt{a}}\right) \quad , \quad \frac{dt}{dv} = \frac{1}{\sqrt{a}}$$

$$\therefore \quad I = \frac{1}{\sqrt{a}} \int_{v_1}^{v_2} \ln(pv + q + \sqrt{v^2 + c})\, dv$$

with $v_1 = \frac{b}{2\sqrt{a}}$, $v_2 = \sqrt{a} + \frac{b}{2\sqrt{a}}$, $p = \frac{\alpha}{\sqrt{a}}$, $q = \beta - \frac{\alpha b}{2a}$, $c = d - \frac{b^2}{4a}$

Then, after a partial integration,

$$I = \frac{1}{\sqrt{a}}\left[v \ln(pv + q + \sqrt{v^2 + c}) - \underbrace{\int v \frac{p + \frac{v}{\sqrt{v^2 + c}}}{pv + q + \sqrt{v^2 + c}}\, dv}_{I_1} \right]\Bigg|_{v_1}^{v_2} \qquad (A-)$$

is obtained.

To reduce I_1 to the integral of a rational function we make a new substitution:

$$u = v + \sqrt{v^2 + c} \text{ , from which}$$

$$v = \frac{u^2 - c}{2u}, \quad \sqrt{v^2 + c} = \frac{u^2 + c}{2u}, \quad \frac{dv}{du} = \frac{u^2 + c}{2u^2}$$

Then, by replacing,

$$I_1 = \int \frac{(1+p)u^4 - 2cu^2 + (1-p)c^2}{2u^2\left[(1+p)u^2 + 2qu + (1-p)c\right]}\, du \quad .$$

A decomposition of the integrand into simple fractions leads to

$$I_1 = \int \frac{u^2 - \frac{2q}{1-p}u + c}{2u^2} du + \underbrace{\int \frac{\frac{2pq}{1-p}u - 2c + \frac{2q^2}{1-p}}{(1+p)u^2 + 2qu + (1-p)c} du}_{I_2} =$$

$$= \frac{1}{2}u - \frac{q}{1-p} \ln u - \frac{c}{2u} + I_2 \qquad \text{(A-2)}.$$

We work on I_2 and we obtain

$$I_2 = \frac{pq}{1-p^2} \int \frac{2(1+p)u + 2q}{(1+p)u^2 + 2qu + (1-p)c} du -$$

$$- \frac{2\left[c(1-p^2) - q^2\right]}{1-p^2} \int \frac{du}{(1+p)u^2 + 2qu + (1-p)c} =$$

$$= \frac{pq}{1-p^2} \ln \left[(1+p)u^2 + 2qu + (1-p)c\right] -$$

$$- \frac{2\sqrt{c(1-p^2) - q^2}}{1 - p^2} \tan^{-1} \frac{(1+p)u + q}{\sqrt{c(1-p^2) - q^2}} .$$

But

$$\ln \left[(1+p)u^2 + 2qu + (1-p)c\right] = \ln \left[2u \left(pv + q + \sqrt{v^2 + c}\right)\right] =$$

$$= \text{constant} + \ln u + \ln \left(pv + q + \sqrt{v^2 + c}\right)$$

and from (A-2)

$$\frac{1}{2}u - \frac{c}{2u} = \frac{u^2 - c}{2u} = v .$$

Then

$$I_1 = v - \frac{q}{1-p^2} \ln \left(v + \sqrt{v^2 + c}\right) + \frac{pq}{1-p^2} \ln \left(pv + q + \sqrt{v^2 + c}\right) -$$

$$- \frac{2w}{1-p^2} \tan^{-1} \frac{(1+p)\left(v + \sqrt{v^2 + c}\right) + q}{w}$$

with

$$w = \sqrt{c(1-p^2) - q^2} .$$

Finally going to (A-1)

$$I = \frac{1}{\sqrt{a}}\left[\left(v - \frac{pq}{1-p^2}\right) \ln\left(pv + q + \sqrt{v^2 + c}\right) - v + \frac{q}{1-p^2} \ln\left(v + \sqrt{v^2 + c}\right) + \right.$$

$$\left. + \frac{2w}{1-p^2} \tan^{-1} \frac{(1+p)\left(v + \sqrt{v^2 + c}\right) + q}{w}\right]\Bigg|_{v_1}^{v_2} \qquad (A-3) \quad .$$