Geof. Int., Vol. 27-3, 1988, pp. 327-350

A SIMPLE METHOD FOR KINEMATIC RAY TRACING IN 3-D HETEROGENEOUS MEDIA, CONTINUOUS AND DISCRETE

J. A. MADRID*

(Received: 19 May, 1987)
(Accepted: 21 January, 1987)

RESUMEN

Se presenta un método sencillo para el trazado cinemático de rayos. El método se basa en la muy conocida "aproximación circular", desarrollada para medios heterogéneos en dos dimensiones. Puede aplicarse a medios tridimensionales tanto continuos como discretos. Los medios en dos dimensiones son un caso particular (perfil), al igual que los medios tridimensionales discretizados, donde las unidades son tetraedros irregulares, dentro de los cuales la velocidad es lineal en las coordenadas. El método usa la forma analítica de la velocidad y sus gradientes. Es rápido, exacto, no requiere mucha memoria ni tiempo de computadora, y permite el diseño y manejo sencillos de heterogeneidades. Se ilustra con ejemplos sintéticos.

ABSTRACT

A simple method for kinematic ray tracing in 3-D heterogeneous media is presented. The method is based on the well known "circular approximation", which holds for 2-D heterogeneous media, and it can be used for both continuous and discrete 3-D media. 2-D media are a particular case (a profile), as well as discrete 3-D media, where the space units are irregular tetrahedrons, inside of which the velocity behaves linearly with the coordinates. The method requires analytical determination of the velocity and velocity gradients; it is fast and accurate, and requires minimum computer time and memory. Also, it is extremely versatile in that it allows easy handling of specific heterogeneities. Synthetic examples are provided.

* *Centro de Investigación Científica y Educación Superior de Ensenada, Baja California.*

INTRODUCTION

The problem of ray tracing has been solved recently in several ways, according to the required usage. In most cases, the so-called ray tracing system of differential equations of the first order (1) is solved. These equations may be presented in different forms, according to the parameter used to describe the raypath. In general

$$\frac{dx_i}{dw} = V^{2-N} p_i$$

$$\frac{dp_i}{dw} = \frac{1}{N} \frac{\partial}{\partial x_i} \frac{1}{V^N} \qquad (1)$$

where pi = $\cos\theta_i/V$, with θ_i the angle of the tangent to the ray at (x, z) with the i-axis, and the velocity V is a function of the coordinates (V = V(x, y, z)).

The nature of dw depends on the value of N: if N = 1, dw = dS (the differential of arch length) if N = 2, dw = VdS, and if N = 0, dw = dT (the differential of travel time). The apparently most efficient choice is N = 2, since if $1/V^2$ is assumed to vary linearly with the coordinates, the solution of system (1) reduces to polynomials of the first, second and third order in w = $\int$VdS for the slownesses, the position and the travel time, respectively (Cerveny, 1986). Several papers have been published that cover both continuous and discrete media: Aric *et al.* (1980), Pereyra *et al.* (1980), Gebrande (1976), Madrid (1985), for example. Most of the methods are lengthy and very time consuming.

Ray tracing is adequate for direct and inverse travel time and synthetic seismograms calculations (*c.f.* Chapman and Drummond, 1982; Cerveny, 1985), as well as in migration of traces and image formation (*c.f.* Carter and Frazer, 1984). The Gaussian beam method (Cerveny *et al.* 1981; Cerveny and Psencik, 1983), for example, has been implemented for 2-D inhomogeneous media (Müller, 1984) that are discretized in such a way that they are composed of a set of "linear" triangular regions. By linear it is understood that the wave velocity behaves as V = Vo + bz′, where the direction of z′ is determined by minus the velocity gradient (−$\underline{b}$).

In this paper, I report a simple, geometrical method to trace raypaths and compute travel times and associate quantities for continuous 3-D media whose wave velocity may be expressed as

$$V = (V(x, y, z) \tag{1.1}$$

with

$$b_x = \frac{\partial v}{\partial x}, \qquad b_y = \frac{\partial v}{\partial y}, \qquad b_z = \frac{\partial v}{\partial z} \tag{1.2}$$

The method is based on the well known "circular approximation" (Aric *et al.*, 1980; Marks and Hron, 1980; Madrid and Traslosheros, 1983; Madrid, 1985). The essentials of this method are that, for every point in the medium the curvature of a ray is given by a suitable combination of $\underline{p}(\underline{x}) = (p, q)$ (the ray slowness) and $\underline{b}(\underline{x}) = (b_x, b_z)$, the velocity gradient. For the curvature of the ray we have

$$k = -p'b \tag{2}$$

where p' is the horizontal component of the slowness, expressed in a system where $\text{grad}(V) = \underline{b}$ is aligned with the negative z'-axis. In two dimensions, the angle of rotation is given by

$$\cos r = -b_z/b$$

$$\sin r = b_x/b$$

and

$$k = pb_z - qb_x \tag{3}$$

For a set of initial conditions (x_0, z_0, p_0, q_0), a circular segment of a raypath may be computed, since the radius is known, and the final values of x, z, p, and q may be calculated. The reiteration of this process traces a ray from a specific point (source) to the surface. The travel time and other parameters are computed and accumulated along the path. The ray remains in the original plane since $b_y = 0$. This is the feature that is exploited to extend the method to 3-D media: in a small neighborhood of (x, y, z), a ray with $\underline{p} = (p, r, q)$ "feels" a velocity which behaves linearly in (x, y, z), so the local path is an arc of a circle contained in a local plane of propagation. If the local path is small enough, the error with respect to the correct path is negligible. The whole path is then composed of many "local" paths, each corresponding to a plane of propagation that may vary according to the behavior of $b = (b_x, b_y, b_z)$.

In the next section, the method is thoroughly explained, and in the following section, synthetic examples of travel time calculations are given. Since the method holds for arbitrary v(x, y, z). It is obvious that an extension to 3-D of the discrete circular approximation becomes a particular case, and it can be easily solved.

THEORY

Consider a medium where the velocity is given by

$$V(x, y, z) = Vo + bz \ , \qquad b < o \tag{4}$$

i.e., $\underline{b} = (o, o, b)$. In such a medium, a ray is totally contained in the plane of propagation defined by the gradient of velocity and the takeoff slowness (Fig. 1a). If the velocity varies in the x and/or y directions, but it is such that b_x = const, b_y = const, a rotation of the cartesian axis of coordinates that aligns the new z-axis (z′) with the direction of $-\underline{b}$ reduces locally the case to the previous one. In this case, however, the plane of propagation is tilted with respect to the vertical (Fig. 1b).

The rotation may be constructed in such a way that the slowness, expressed in the new system ($\underline{p}' = \underline{p}$, due to invariance with respect to rotations) is totally contained in the plane of propagation. Then, the y′− component of the slowness vector is null. This is done as follows:

A unit vector in the direction of the z′− axis is given by:

$$k_1' = \frac{-b_x}{b}$$

$$k_2' = \frac{-b_y}{b}$$

$$k_3' = \frac{-b_z}{b}$$

i.e.

$$k' = (k_1', k_2', k_3') \tag{5.1}$$

A unit vector perpendicular to the plane that contains the ray path is obtained by taking the vector product

$$\hat{j} = \hat{k}' \times \underline{p}v$$
$$= (j'_1, j'_2, j'_3) , \qquad (5.2)$$

(a)

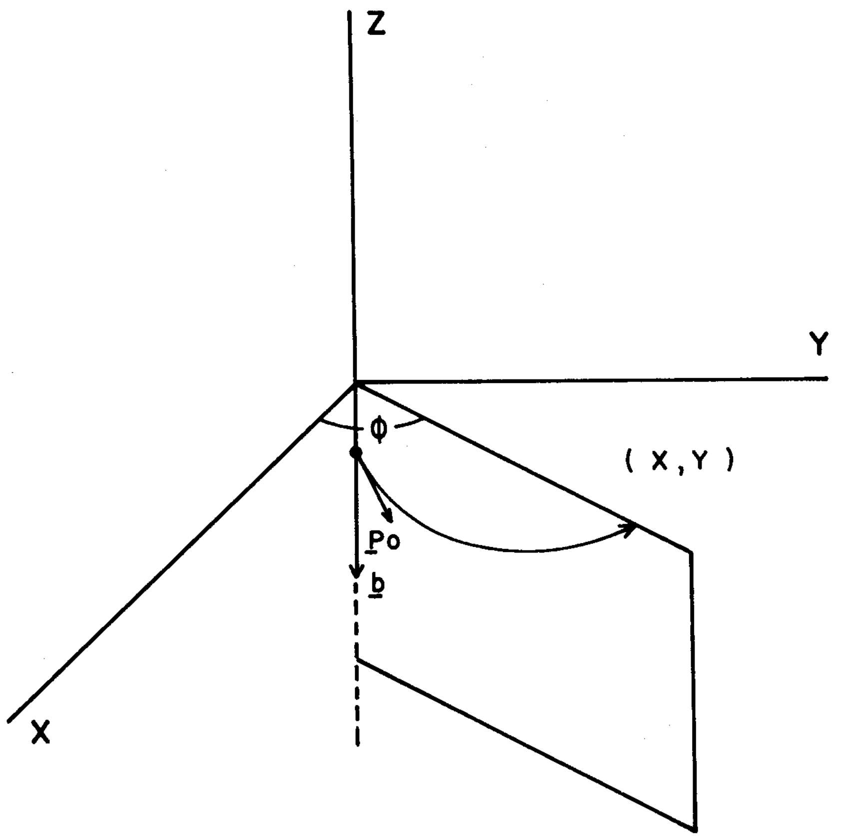
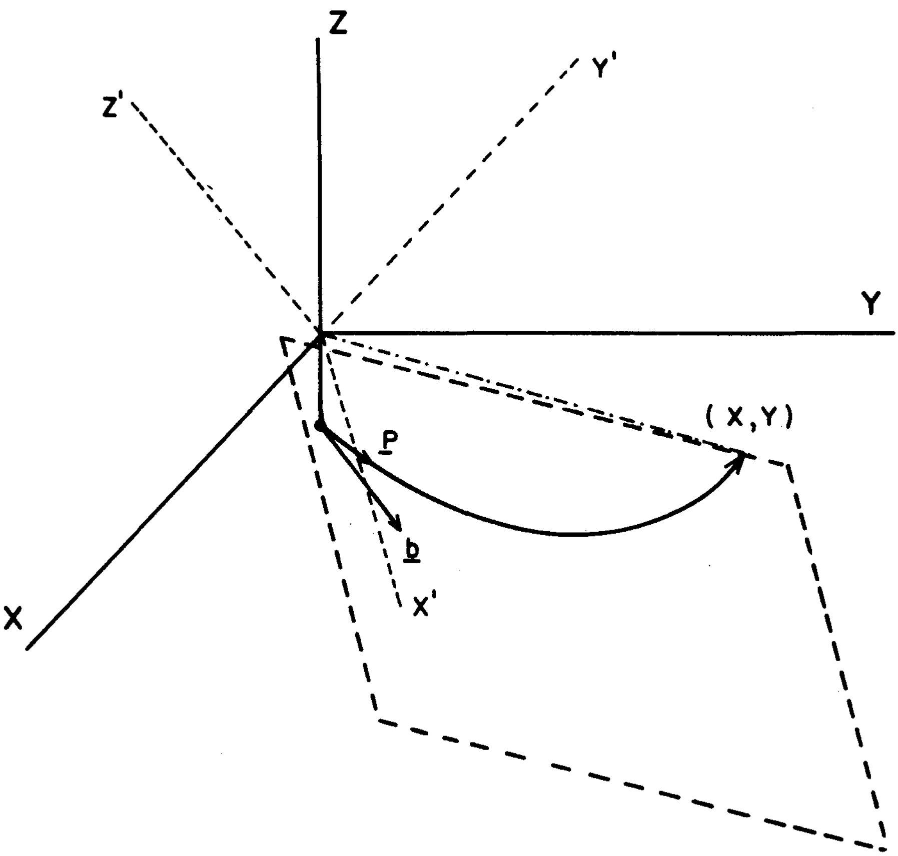


Fig. 1. *(a)* A ray with initial slowness p in a medium with velocity V = Vo+ bz remains in the plane defined by the normal to the slowness p and to the velocity gradient b = (b_x, b_y, b_z). The horizontal slowness p is conserved. *(b)* If b_x, b_y, $b_z \neq 0$ the ray remains in the plane defined by p and b, but this is no longer vertical and the horizontal slowness p is not conserved.

(b)



since pv is the unit slowness vector; finally, the right-handed system of coordinates is completed simply by another vector product:

$$\hat{i}' = \hat{j}' \times \hat{k}'$$

$$= (i_1', i_2', i_3') \tag{5.3}$$

The operator formed by writing (5.3), (5.1) and (5.2) (hereafter referred as (5)), as the rows of a matrix is a rotation that transforms the (x, y, z) system of coordinates into the system (x′, y′, z′). Since it is orthogonal, its inverse is simply the transpose matrix.

$$R = \begin{pmatrix} i'_1 & i'_2 & i'_3 \\ j'_1 & j'_2 & j'_3 \\ k'_1 & k'_2 & k'_3 \end{pmatrix} \tag{5}$$

$$R^{-1} = R^T$$

When the velocity gradients are constant, one such construction is enough, since, as discussed in the first paragraph, the whole ray path remains in the same plane. This is not the case if the velocity of propagation is a more general function of the coordinates.

Let us assume that the velocity is given by (1.1). The gradient of velocity may change from point to point, and the raypath will be a curve in the 3-D (x, y, z) space that is not contained in a specific plane. It is intuitive, however, that the whole ray path may be divided into a finite number of circular segments, as illustrated with circle arcs in Fig. 2. Thus, the construction of the whole ray path may be achieved by sequentially constructing the required circular segments through a method such as those discussed in Gebrande (1976) or Madrid (1985). This case is not different from the case of a constant gradient, except that the construction of the local system (5.1, 5.2, 5.3) must be done for every segment along the ray path.

In the two dimensional case, the determination of a segment is immediate. The only feature to check is that the whole path (Fig. 3) remain inside the region (Madrid, 1985). If this is not so, the circular segment must be interpolated to the proper boundary. The continuation of the ray path then requires the application of Snell's law.

The 3-D case is, of course, more complex. The division of the whole medium may be done in a variety of ways, but in any case, the different regions that conform the medium must be in welded contact with each other. Perhaps, the simplest way is to use irregular blocks. The important feature required to define a ray displacement as

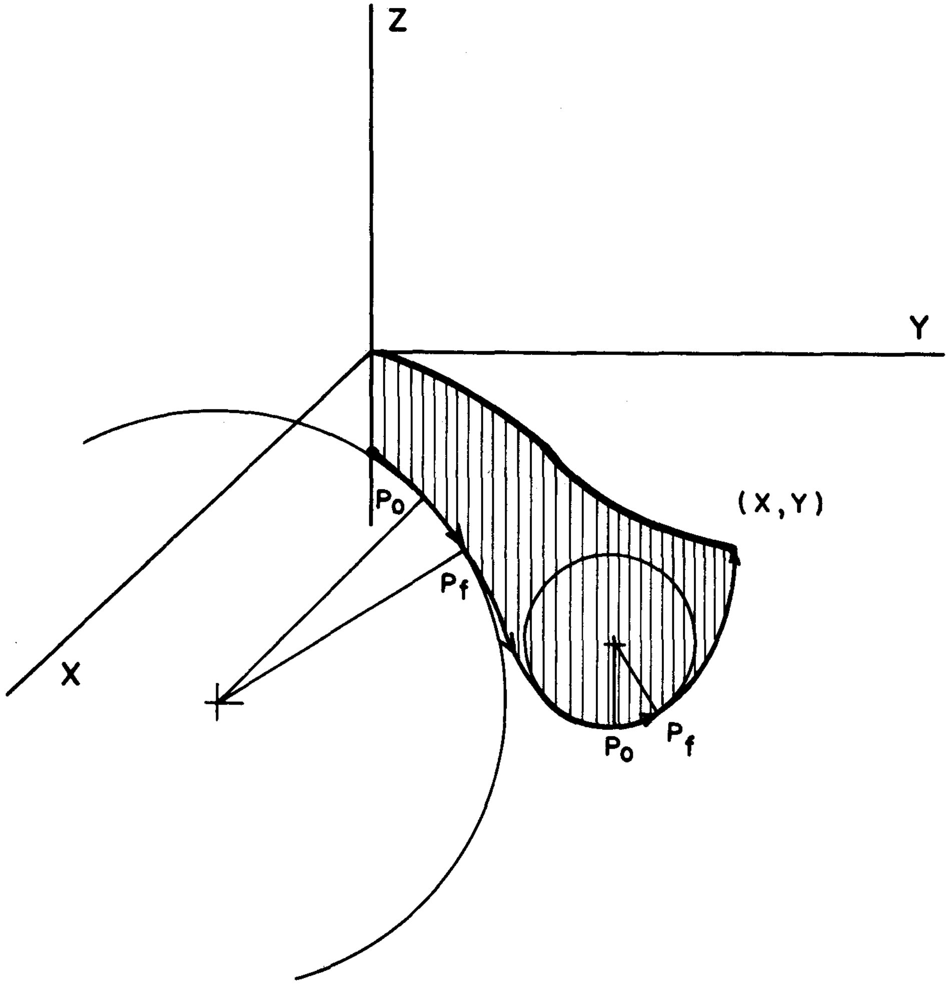


Fig. 2. A whole raypath is composed of a number of "local arcs of circles". The radiuses of the osculating circles may vary according to the variation of the velocity. The heavy line above is the projection of the ray on the surface z = 0.

"good" is that it does not cross any boundary face, *i.e.*, both the starting as well as the final points of the segment are inside the region. If this requirement is not satisfied, the intercept of the local osculating circle with the appropriate face of the cube must be determined and Snell's law applied, just as in the 2-D case. A simple way to do this is as follows: once the local plane of propagation and the local circle have been constructed, the intercepts of the plane with the sides of each face of the cube can be easily computed. For the sides of each face, there can be only two such intercepts. This is shown in Fig. 4, where the vertices of a face are numbered 1, 2, 3, 4, and the intercepts are named I_1 and I_2. Calling, for example, $\underline{r}_1$ and $\underline{r}_2$ the vectors defining the vertices 1 and 2, we have:

$$r = \underline{r}_1 + \lambda(\underline{r}_2 - \underline{r}_1) \tag{6}$$

Since the rotation that aligns the normal to the plane of propagation with the y′ axis is known (5), the whole cube is rotated, and (6) becomes:

$$\begin{aligned} x' &= x_1' + \lambda(x_2' - x_1') \\ y' &= y_1' + \lambda(y_2' - y_1') = yp' \\ z' &= z_1' + \lambda(z_2' - z_1') \end{aligned} \tag{7}$$

λ can be determined from the second equation:

$$\lambda = \frac{yp - y_1'}{y_2 - y_1} \tag{8}$$

x and z′ are then easily determined by direct substitution of (8) in (7). The problem then has been reduced to a local two dimensional problem, similar to the one explained in Madrid (1985). The only difference is that, in that case, there were always four sides to be checked, whereas in the present case the number of sides may vary from 3 to 6 (Fig. 5).

The final step for deciding if a displacement is bad or good is simple if a second rotation is implemented, this time, in the 2-D space of the plane of propagation. This rotation aligns the x″ axis with the direction of the line defined by the intercepts. In this new (x″, z″) system, the osculating circle intercepts the line at the points

$$xc'' \pm \Delta x'' \tag{9}$$

where

$$\Delta x'' = \pm (R^2 - D^2)^{1/2} \tag{10}$$

and D is the perpendicular distance from the center of the circle to the line L (defined by I_1 and I_2). The last step consists of eliminating the solutions that do not comply with the direction of the ray or that fall outside the intercepts of the plane of propagation with the face of the cube. Only one solution is left, say (x_f'', z_f''). The inverse rotation from the (x'', z'') system produces the point

$$(x_f', y_p', z_f') \tag{11}$$

and the inverse operator of (5), applied to (11) gives the final point

$$(x_f, y_f, z_f) \tag{12}$$

To complete the algorithm, the travel time between points p_o and p_f must be calculated, as well as the final slowness components. The slowness is easily determined in either the normal system of coordinates or the local system, but it is easier in the latter through

$$q_f' = \frac{x_f' - x_o'}{R\, V_f}$$

since inside this system $p' =$ const and $r' = 0$, the inverse operator to (5) then gives

$$\begin{pmatrix} p_f \\ r_f \\ q_f \end{pmatrix} = R^{-1} \begin{pmatrix} p' \\ o \\ q_f' \end{pmatrix} \tag{13}$$

the travel time is then

$$\Delta t = \frac{1}{b} \left[\ln \frac{1 + qv}{pv} \right]_o^f \tag{14}$$

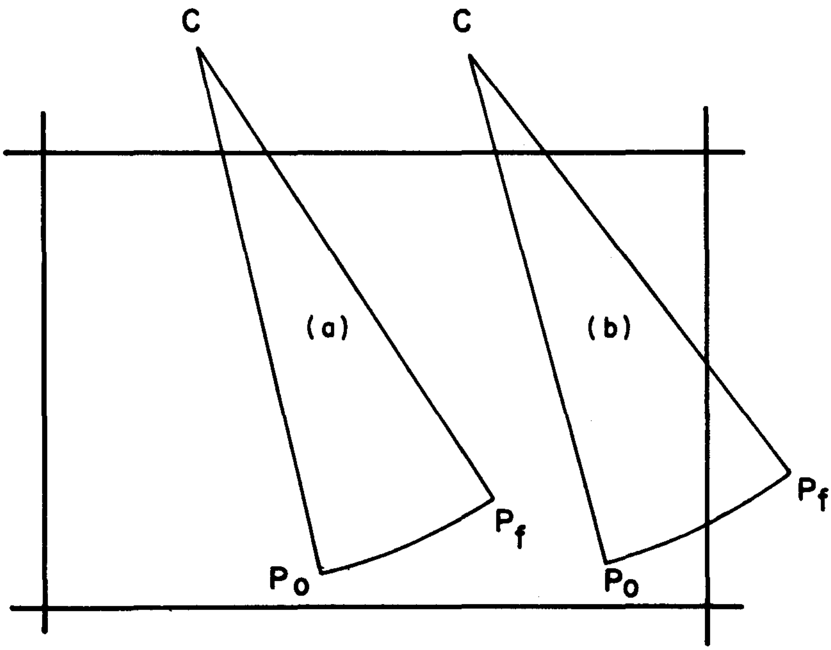


Fig. 3. *(a)* A "good" segment. Both, the initial and the final points are inside the same region. *(b)* A "bad" segment. The final point must be interpolated to the proper side of the region; there, Snell's law must be used to determine if the ray is reflected or refracted, as well as the new direction of the ray.

Summation of (14) for each segment along the whole path yields the total travel time. Other quantities are derived from T, X, Y, p_f, r_f, q_f.

For discrete media, the procedure explained is as follows. The media are divided in blocks that in turn are divided in tetrahedrons defined by the vertices of the cubes. Each cube is formed by six tetrahedrons (Fig. 6), inside each one of them the velocity behaves as

$$V = Vo + b_x\,x + b_y\,y + b_z\,z. \qquad (15)$$

The four constants (Vo, b_x, b_y, b_z) in (15) are easily computed using the rule of Kramer, since at the vertices of each tetrahedron

$$V = Vo + b_x\,x_i + b_x\,y_i + b_z\,z_i\,, \qquad i = 1, 2, 3, 4$$

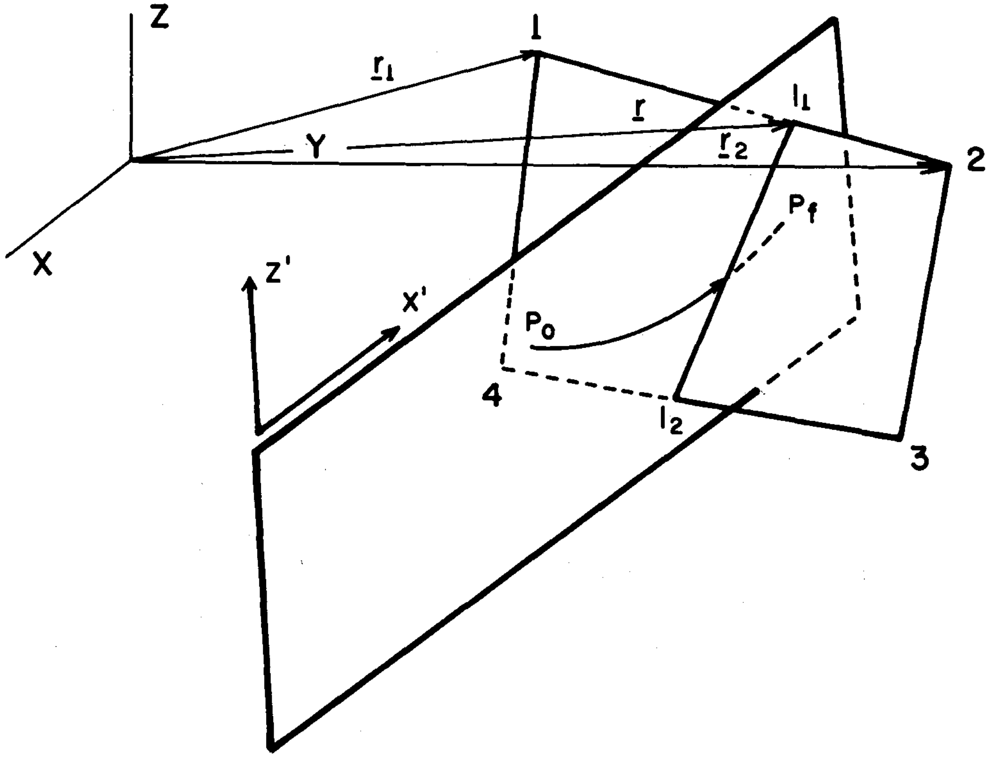


Fig. 4. A ray striking on one of the faces of a cubic region. The vertices of the face are numbered 1, 2, 3, 4. The local plane of propagation is intersected by the sides 1-2 and 3-4 at points I_1 and I_2. The interpolation to the correct final point is performed more easily in the system of the plane of propagation (x', z').

The formulae developed in the present section are valid in this case, the only difference is in the final point, which, for each tetrahedron, is computed in only one step, *i.e.*, solving for the intercept of the local circle with the appropriate face of the tetrahedron. The final point thus obtained becomes the initial point of a new tetrahedron. Of course, for continuous media where the velocity gradient is constant, this procedure may be used too.

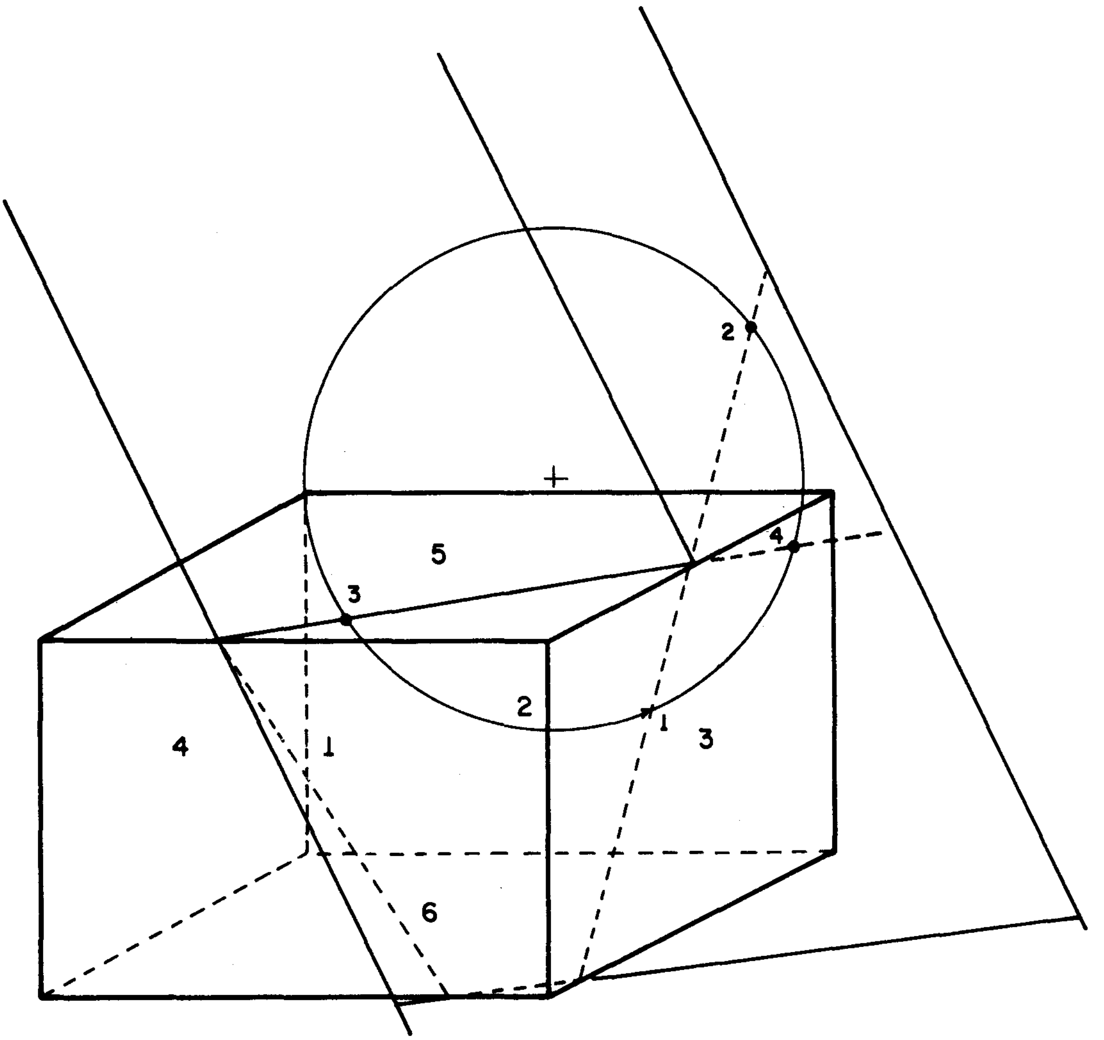


Fig. 5. An example of the variation of the number of faces that are cut by a local plane of propagation. Here the plane intercepts only four out of the six faces. If the inclination of the plane is changed, it may cut more or less faces. In this example, the important faces are 1(front), 3(right), 5(above), 6(below), but solutions exist only for faces 3 and 5 (points 1, 2, 3, 4). Two of the possible solutions (2, 4) fall outside the intercepts, and are consequently eliminated. The other one (3) does not comply with the direction of the ray.

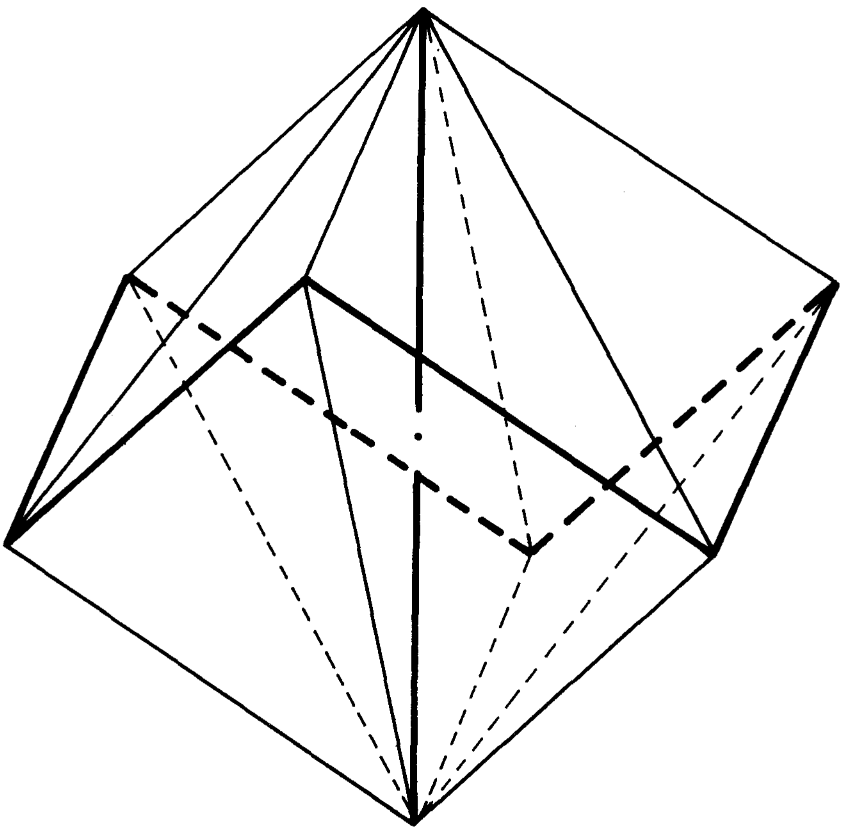


Fig. 6. Any cube may be divided in 6 tetrahedrons.

EXAMPLES

The following examples are very simple and have little geophysical interest, but fully illustrate the capabilities of the method.

The first two examples differ only in that the velocity gradient has, correspondingly 1 (b_z), and 2(b_y, b_z) components, thus they are equivalent after an adequate rotation of the coordinate axes. However, in the x-y plane (z = o), the difference is noticed in the shape of the isochrones or equal travel time curves at the surface.

Figure 7 shows the case $b_x = b_y = 0$, $b_z \neq 0$. Profiles are shown in part (a), at three different azimuths. The rays are arcs of circles contained in vertical planes and the resulting profiles are straight. In part (b), thirty rays with the same vertical take-off angle, but different azimuths are plotted, and a vertical sight of these rays is shown in part (c). In Fig. 8(a) and (b) the same curves are illustrated for the case $b_y \neq 0$, $b_z \neq 0$, $b_x = 0$. It is evident here that the isochrones become circles elongated in a direction opposite to $(b_x^2 + b_z^2)^{1/2}$, *i.e.* along the y-axis. The rays are circles contained in planes that are tilted so that the normal to each plane is given by pxb. This example is equivalent to the more general $b_x \neq 0$, $b_y \neq 0$, $b_z \neq 0$, after a rotation in the (x, y) plane that aligns the horizontal component of the gradient with a new y-axis.The interesting feature of this example is that the profiles are not straight anymore. That is, the rays that correspond to a vertical plane Φ = const at the source do not emerge in a straight line (Fig. 8(b)) (a profile) at the surface. In Fig. 8(c), 30 rays at azimuths spanning 2π radians around the source are illustrated; here, the clustering of the rays in the direction of the velocity gradient is very noticeable. Fig. 8(d) is a vertical view of these rays.

(a)

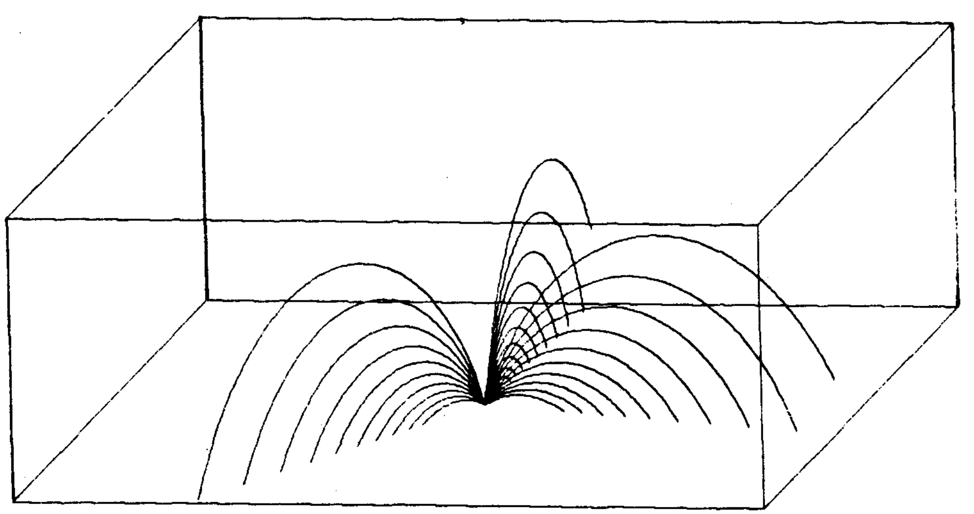
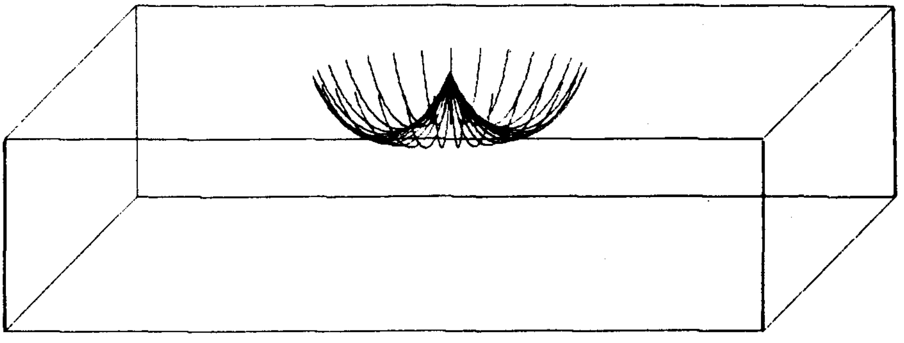
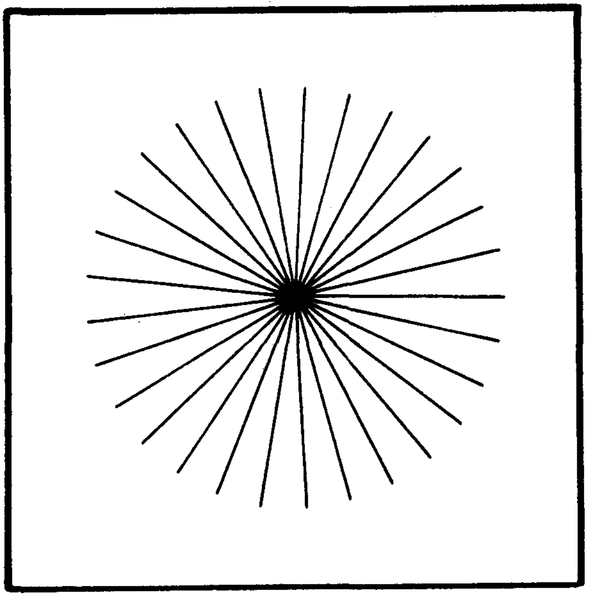


Fig. 7. *(a)* A medium with $b_x = b_y = 0$, $b_z \neq 0$. Three profiles at different azimuths are shown. Two of the rays leave the model through lateral sides. *(b)* 30 rays with equal takeoff angle spanning 2¶ radians around the source. *(c)* The rays of *(b)* as seen from above.

(b)



(c)



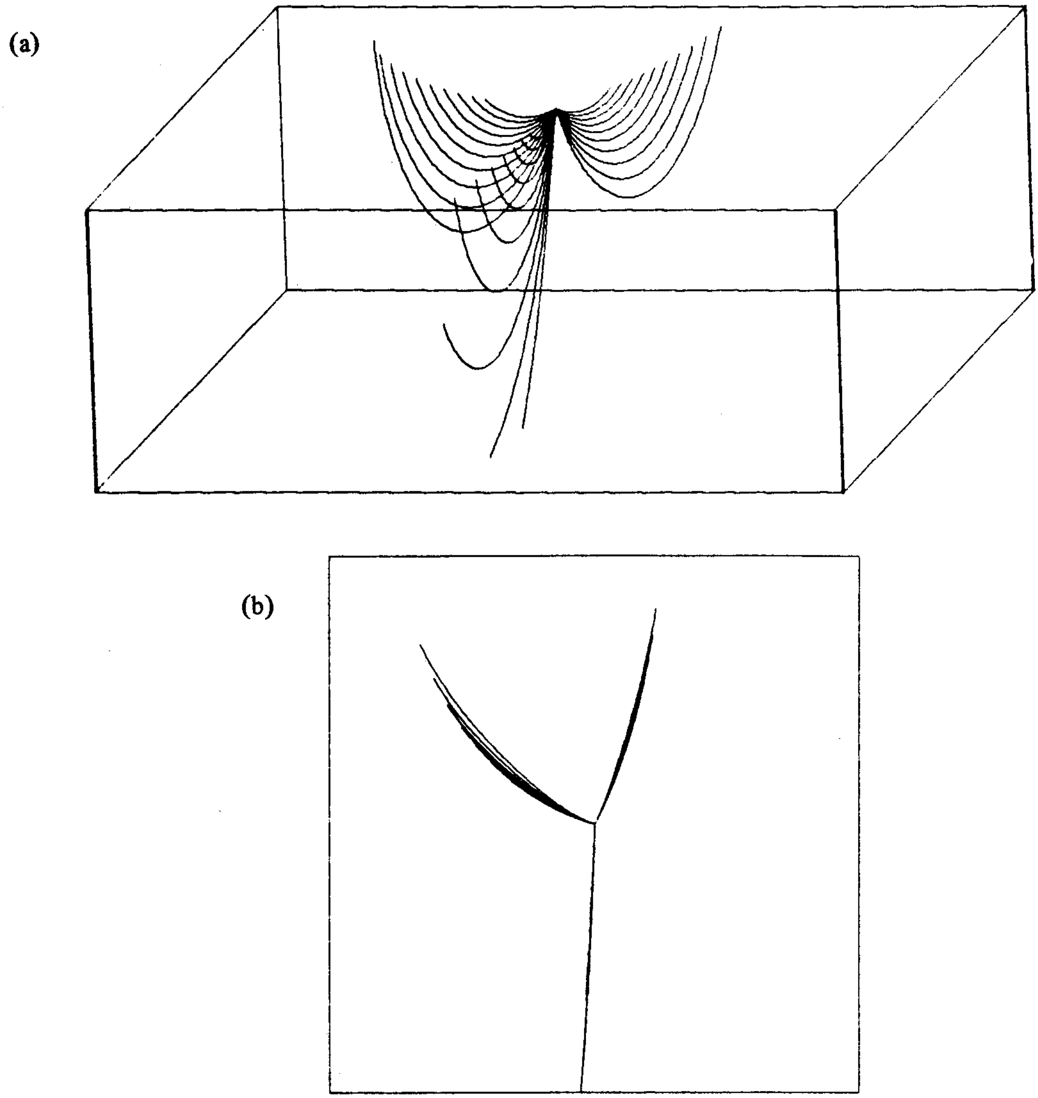
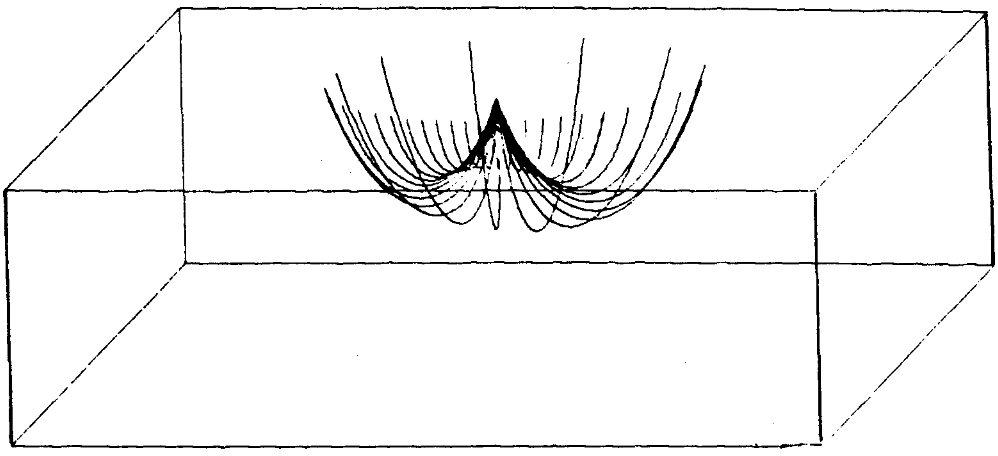
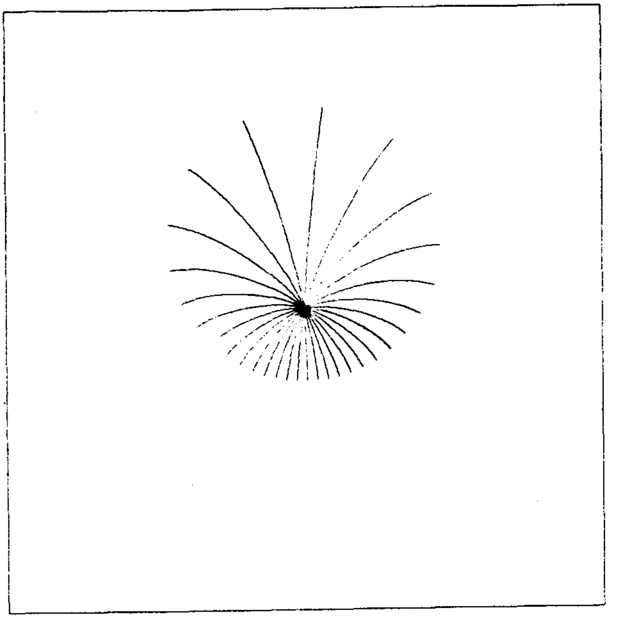


Fig. 8. *(a)* Same region as in Fig. 6, but $b_y \neq 0$. *(b)* The vertical sight of the profiles shows that they are not straight anymore. *(c)* Thirty rays at equal takeoff angle showing divergence in the direction of decreasing velocity and clustering in the opposite direction. *(d)* Vertical sight of *(c)*.

(c)



(d)



The fourth example, shown in Fig. 9, includes an inclined discontinuity that causes that rays in a certain angular range be reflected or refracted. Critically refracted rays are not illustrated, but they could easily be drawn. Three reflections in the wall are easily spotted.

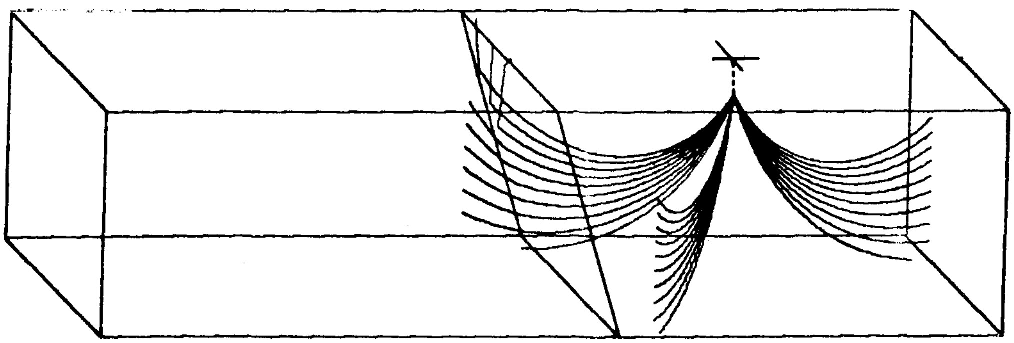


Fig. 9. Two media with vertical gradient separate by an inclined discontinuity. Three different shooting directions are shown, two of them yield rays leaving the model in vertical walls, the other shows three reflected rays.

The example shown in Fig. 10 is the simplest representation of a graben. Three blocks have been used where the velocity varies with depth, but it is greater in the middle one. In the figure, two refractions appear, one in each inclined fault. The effect of the refraction in the fault at the rear is noticed in the change of direction of the profile (see the dotted line).

The last example (Fig. 11) corresponds to a model that includes four layers with discontinuities. This example is illustrative of the versatility and simplicity with which models are handled when worked out using analytical functions to define the velocity. For example, if one wishes to model a bell shaped anomaly within a 'normal' medium ($V = Vo + bz$), the following set of definitions may be used:

Block 1: $$V(z) = Vo + bz, \quad b_x = b_y = 0 \quad b_z = b \tag{16.1}$$

Block 2: $$V = V(z) + \Delta V_2(z)\,\mathrm{sech}(A\,|r - r_0|)$$

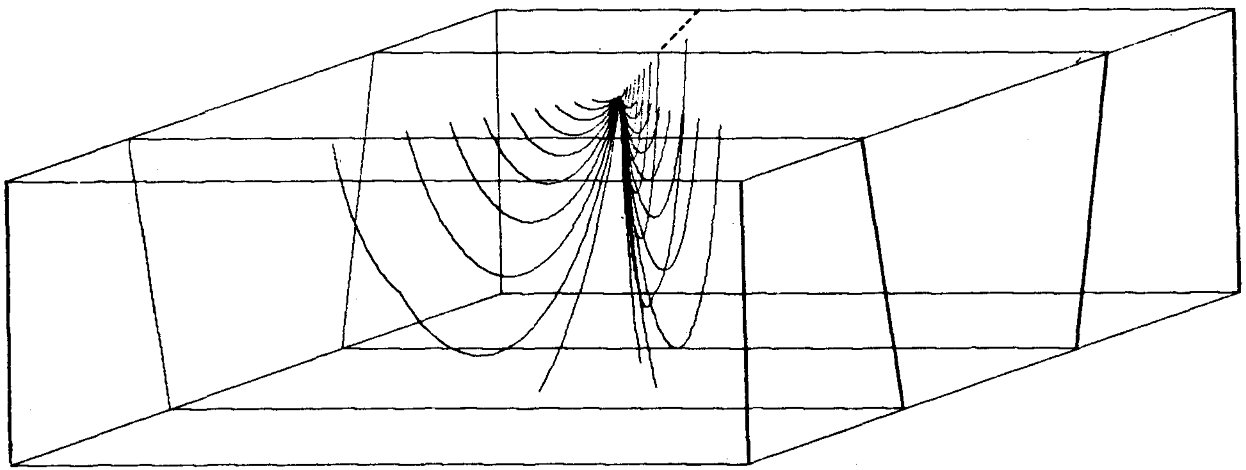


Fig. 10. A simple representation of a graben. The velocity of propagation in the middle region is greater. Three different azimuths are shown. Note the change of direction of the profile due to refraction on the rear fault.

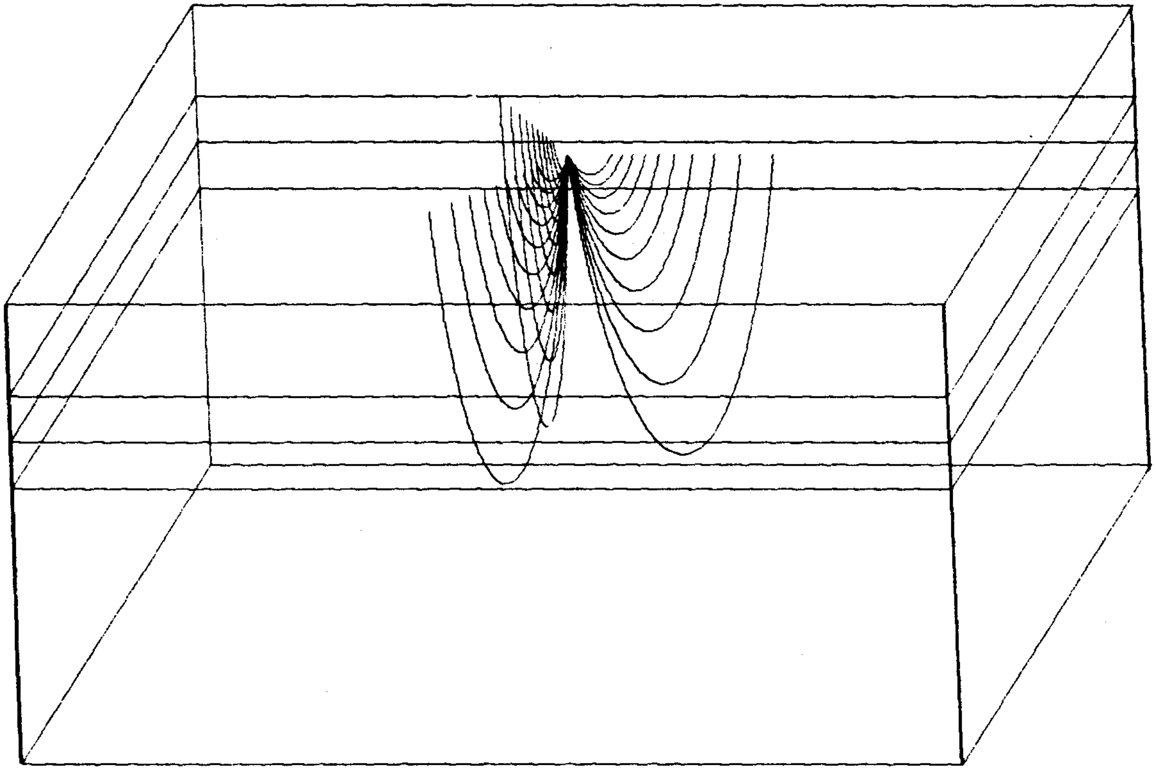


Fig. 11. A four layers velocity model with three shooting directions. In this example, the same velocity law holds through the four regions, so that the rays are completely circular.

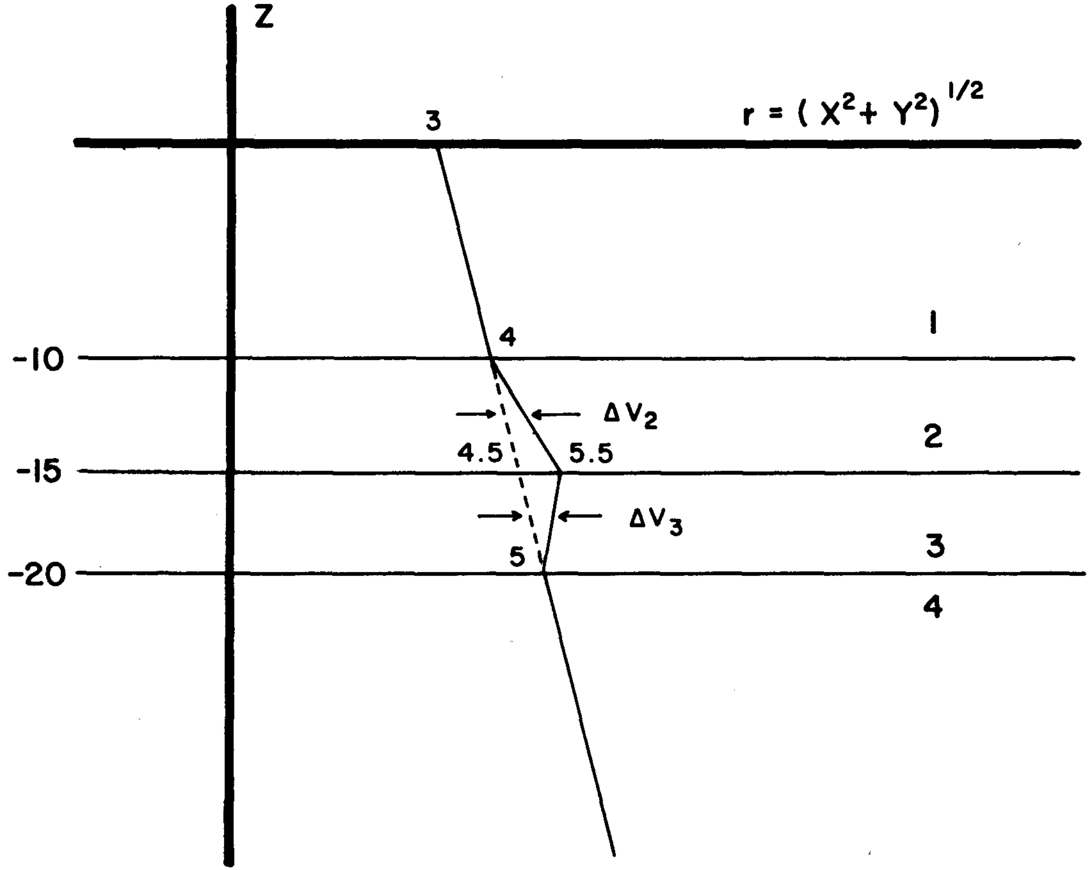


Fig. 12. A section of a possible representation for a "bell shaped" velocity anomaly as given by equations (15.1), (15.2) and (16), with $r = r_0$.

$$b_x = \Delta V_2(z)\,\mathrm{sec\,h}(A\,|r - r_0|)\,\tanh(A\,|r - r_0|)\,\frac{A(x - x_0)}{|r - r_0|}$$

$$b_y = \Delta V_2(z)\,\mathrm{sec\,h}(A(r - r_0)\,\tanh(A\,|r - r_0|)\,\frac{A(y - y_0)}{|r - r_0|} \qquad (16.2)$$

$$b_z = b + \frac{\partial}{\partial z}\,(\Delta V_2(z)\,\mathrm{sec\,h}(A\,|r - r_0|)$$

where $|r - r_0| = ((x - x_0)^2 + (y - y_0)^2)^{1/2}$, and (x_0, y_0)

is the center of the anomaly.

Block 3: same as Block 2, with $\Delta V_3(z)$ instead of $\Delta V_2(z)$

Block 4: same as Block 1.

In particular, if $r = r_0$, and

$$\Delta V_2(z) = -.2(z + 10), \qquad -10 > z > -15$$

$$\Delta V_3(z) = .2(z + 20), \qquad -15 > z > -20 \qquad (16.3)$$

the model corresponds to Fig. 12. If $r \neq r_0$, the amplitude of ΔV decreases.

If a discontinuity must be considered, it is enough to add or(and) substract a constant in the corresponding block(s).

BIBLIOGRAPHY

ARIC, K., R. GUTDEUTCH and A. SAILER, 1980. Computation of travel times and rays in a medium of two-dimensional velocity distribution. *Pure and Applied Geophys., 118,* 796-805.

CARTER, J. A. and L. N. FRAZER, 1984. Accomodating lateral velocity changes in Kirchoff migration b means of Fermat's principle, *Geophysics, 49,* 1, 46-55.

CERVENY, V., M. M. POPOV and I. PSENCIK, 1982. Computation of wave fields in inhomogeneous media- Gaussian beam approach. *Geophys. J. R. Astr. Soc., 70,* 109-128.

CERVENY, V. and I. PSENCIK, 1983. Gaussian beams in two-dimensional elastic inhomogeneous media. *Geophys. J. R. Astr. Soc., 72,* 417-433.

CERVENY, V., 1985b. Ray synthetic seismograms for complex two-dimensional and three-dimensional structures. *J. Geophys., 58,* 2-26.

CERVENY, V., 1986. Ray tracing algorithms in three dimensional laterally varying layered structures. *In:* Tomography in seismology and Exploration Seismics, G. Nolet, Ed., D. Reidel Publ. Co. The Netherlands.

CHAPMAN, C. H. and R. DRUMMOND, 1982. Body wave seismograms in inhomogeneous media using Maslov Asymptotic Theory. *Bull. Seism. Soc. Am., 72,* S277-S317.

GEBRANDE, H., 1976. A seismic-ray tracing method for two dimensional inhomogeneous media. *In:* Explosion Seismology in Central Europe, Eds. P. Griese, C. Prodehls and A. Skin, Springer-Verlag, Berlin, 162-167.

MADRID, J. y C. TRASLOSHEROS, 1983. Un modelo sísmico preliminar lateralmente heterogéneo del campo geotérmico de Cerro Prieto, Baja California Norte. *Geofís. Int., 22,* 4, 389-417.

MADRID, J. A., 1985. Travel times and ray paths for continuous media. *Geofís. Int., 24,* 3, 439-458.

MADRID, J. A., 1986. New formulae for linear travel time inversion in 2-D heterogeneous media. Theory and results. *Geofís. Int., 25,* 3, 361-382.

MARKS, L. W. and F. HRON, 1978. Personal communication.

MULLER, G., 1984. Efficient calculation of Gaussian-beam seismograms for two-dimensional inhomogeneous media. *Geophys. J. R. Astr. Soc. 79,* 153-166.

PEREYRA, V., W. H. K. LEE and H. B. KELLER, 1980. Solving two-point seismic-ray tracing problems in a heterogeneous medium. *Bull. Seism. Soc. Am., 70,* 1, 79-99.