Geofísica Internacional (1992), Vol. 31, Num. 2, pp. 187-191

Ray tracing in the troposphere for GPS satellite signals
Part I

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RESUMEN
En este trabajo proponemos un algoritmo adecuado para el trazado de rayos en la troposfera para las señales del satélite GPS (Global Positioning System). Se usan datos aerológicos (presión, temperatura y humedad relativa) para diferentes altitudes, las cuales se pueden obtener mediante radiosondeos. Como el camino de un rayo depende de las condiciones aerológicas en el tiempo y el espacio, y no es vertical cuando llega a la troposfera, suponemos que la refractividad es una función exponencial de la altitud y que sus valores se distribuyen en superficies esféricas. Más aún, suponemos que la troposfera es estática, estratificada esféricamente y concéntrica con la Tierra. Como resultado, obtuvimos que el camino óptico es mayor que el geométrico. Este hecho se debe considerar para mejorar el posicionamiento geodésico.

PALABRAS CLAVE: troposfera, trazado de rayos, señales del satélite GPS.

ABSTRACT
In this paper, we propose a suitable algorithm to compute ray tracing in the troposphere for GPS (Global Positioning System) satellite signals. We use aerological data (pressure, temperature and relative humidity) for different altitudes, which can be obtained by radiosondes. Because a ray path depends on aerological conditions in time and space, and is not vertical when it reaches the troposphere, we assume that the refractivity is an exponential function of altitude and its values are distributed on spherical surfaces. Furthermore, we assume that the troposphere is static, spherically stratified and concentric with the Earth. As a result, we obtain that the optical path is greater than the geometric one. This fact must be considered to improve geodetic positioning.

KEY WORDS: troposphere, ray tracing, GPS satellite signals.

INTRODUCTION

Signals emitted by GPS satellites traverse the ionosphere and troposphere to be received at a moving or fixed station. Signals suffer a delay due to the refractive effects as they traverse an inhomogeneus and stratified medium. In this model we use Snell's law for refraction and consider that the troposphere presents spherical stratification and is concentric with the Earth.

The refraction relation for the troposphere is:

$$n\, r \sin \phi = cte \qquad (1)$$

where

- n is the refractive index at any point.
- r is the distance between this point and the center of the Earth.
- ϕ is the angle between the radial direction at any point and its tangent.

Figure 1 shows this situation.

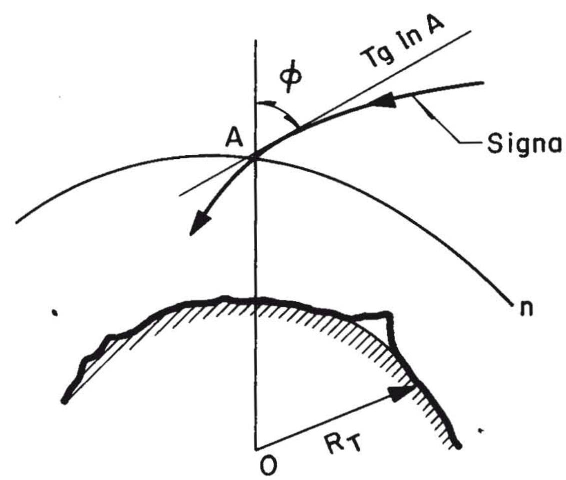

Fig. 1

In this work we utilize N, the refractivity of the medium, instead of n, the refractive index. Furthermore the refraction index only depends on the altitude. Thus, if we have the refractive index for any altitude we can know the signal direction using equation (1). If we know the wave arrival angle in the troposphere, where the refraction index is known too, we can use an iterative method to obtain the ray path.

THEORETICAL DEVELOPMENT

For the ray trajectory we use the geometry shown in Figure 2.

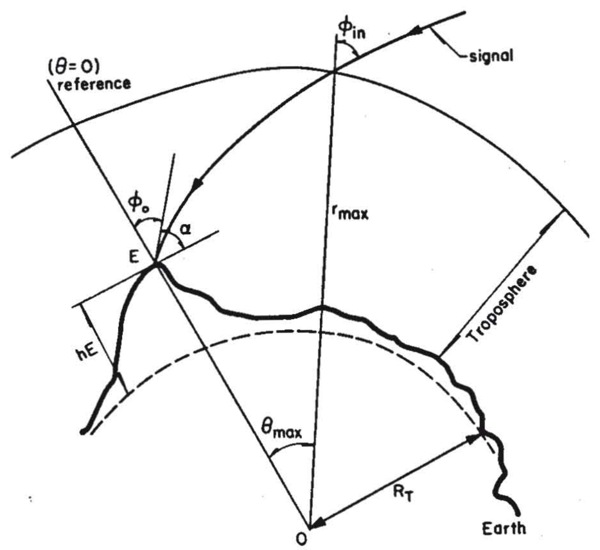

Fig. 2

where

R_T is the Earth's radius.

h_E is the station altitude with respect to sea level.

ϕ_{in} is the angle between radial direction and the tangent of the arriving signal.

ϕ_o is the angle between radial direction and the tangent at a point at the receiving station.

r_{max} is the altitude of troposphere.

For two different positions r_k and r_{k+1} at two points in the ray trajectory (see figure 3) we use a polar coordinate frame (r, θ) and the ray trajectory can be written as $r = r(\theta)$.

For any h_k altitude (at a point k) above sea level, the length r_k will be:

$$r_k = R_T + h_k \tag{2}$$

If the angular position, θ_k, decreases an amount $\Delta\theta = \varepsilon$, the new position will be:

$$r_{k+1} = R_T + h_{k+1} \tag{3}$$

where the arc length at the sea level is $\Delta_s = R_T . \varepsilon$.

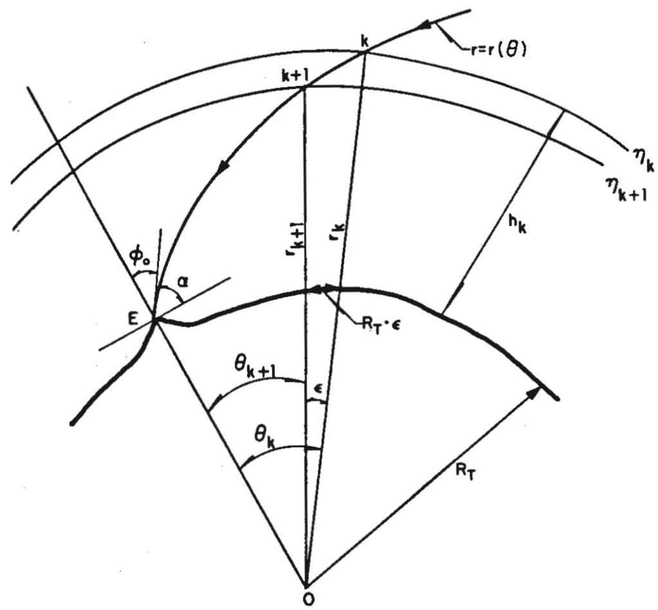

Fig. 3

The ray trajectory can be written as:

$$r = r(\theta) = F(\theta) \tag{4}$$

For k + 1 the position will be $F(\theta + \varepsilon)$, and

$$r_{k+1} = F(\theta + \varepsilon) \tag{5}$$

For differential values of ε we can expand equation (5) in a Taylor series as:

$$F(\theta + \varepsilon) = F(\theta) + \varepsilon \frac{dF(\theta)}{d\theta} + \ldots + \frac{\varepsilon^n}{n!} \frac{d^nF(\theta)}{d\theta^n} + \ldots \tag{6}$$

Taking only three terms of equation (6) and including it in equation (4) we get:

$$r_{k+1} = r_k + \varepsilon \left[\frac{dr}{d\theta}\right]_k + \frac{\varepsilon^2}{2!} \left[\frac{d^2r}{d\theta^2}\right]_k \tag{7}$$

If we have $(dr/d\theta)$ k and $(d^2r/d\theta^2)$ k it is possible to obtain, by an iterative method, Snell's law [eq. (1)] for two correlative positions:

$$n_o r_o \sin \phi_o = n r \sin \phi \tag{8}$$

Considering

$$\phi_o = \frac{\pi}{2} - \alpha$$

we get

$$\sin \phi_o = \cos \alpha$$

and

$$\sin\phi_o = \frac{n}{n_o} r_o \sin\phi \tag{9}$$

and detecting r:

$$r = \frac{n_o}{n} r_o \frac{\operatorname{cosec}\phi}{\sec\alpha} \tag{10}$$

Which for point k will be:

$$r_k = \frac{n_o}{n} r_o \frac{\operatorname{cosec}\phi_k}{\sec\alpha} \tag{11}$$

We can obtain $(dr/d\theta)$ k and $(d^2r/d\theta^2)$ k considering:

i) nk = const for each stratification

ii) nk ≠ const for each stratification.

MODEL I

Now we analyze a n = const case:

$$\left[\frac{dr}{d\theta}\right]_k = \frac{n_o r_o}{n_k \sec\alpha}\left[\frac{\cos\phi_k}{\sin^2\phi}\left(-\frac{d\phi}{d\theta}\right)\right]_k \tag{12}$$

from this we can prove that $(d\phi/d\theta)$ k = -1, and

$$\left[\frac{dr}{d\theta}\right]_k = \frac{n_o r_o}{n_k \sec\alpha}\operatorname{cosec}\phi_k\left[\operatorname{cosec}^2\phi_k - 1\right]^{1/2} \tag{13}$$

in which, including equation (11) we get:

$$\left[\frac{dr}{d\theta}\right]_k = r_k\left[\operatorname{cosec}^2\phi_k - 1\right] \tag{14}$$

For the second derivative:

$$\left[\frac{d^2r}{d\theta^2}\right]_k = \left[\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right)\right]_k = \frac{d}{d\theta}\left[r_k\left(\operatorname{cosec}^2\phi_k - 1\right)^{1/2}\right] \tag{15}$$

which is

$$\left[\frac{d^2r}{d\theta^2}\right]_k = \left(\operatorname{cosec}^2\phi_k - 1\right)^{1/2}\left(\frac{dr}{d\theta}\right)_k + r_k \operatorname{cosec}^2\phi_k\left(\frac{d\phi}{d\theta}\right)_k \tag{16}$$

then, substituting in to equation (14) and considering $(d\phi/d\theta)$ k = -1, equation (16) can be written

$$\left[\frac{d^2r}{d\theta^2}\right]_k = \left(\operatorname{cosec}^2\phi_k - 1\right)^{1/2} r_k\left(\operatorname{cosec}^2\phi_k - 1\right)^{1/2} - r_k \operatorname{cosec}^2\phi_k \tag{17}$$

ordering terms and simplifing,

$$\left[\frac{d^2r}{d\theta^2}\right]_k = -r_k \tag{18}$$

Thus, equation (7) will be

$$r_{k+1} = r_k + \varepsilon r_k\left[\left(\operatorname{cosec}^2\phi_k - 1\right)^{1/2} - \frac{\varepsilon}{2}\right] \tag{19}$$

This equation can be utilized for iterative calculus in ray tracing when the troposphere is considered spherically stratified and each layer has constant refractivity.

MODEL II

Now we utilize statement ii), say, the refractive index n ≠ constant. Then the first derivative $(dr/d\theta)$ k is

$$\left[\frac{dr}{d\theta}\right]_k = r_k\left[\left(\operatorname{cosec}^2\phi_k - 1\right)^{1/2} - \frac{1}{n_k}\left(\frac{dn}{d\theta}\right)_k\right] \tag{20}$$

where

$$\left(\frac{dn}{d\theta}\right)_k = \frac{r_k\left(\operatorname{cosec}^2\phi_k - 1\right)^{1/2}}{1 + \frac{r_k}{n_k}\left(\frac{dn}{dr}\right)_k}\left(\frac{dn}{dr}\right)_k \tag{21}$$

The second derivative, $(d^2r/d\theta^2)$ k, is

$$\left[\frac{d^2r}{d\theta^2}\right]_k = \frac{d}{d\theta}\left\{r_k\left[\left(\operatorname{cosec}^2\phi_k - 1\right)^{1/2} - \frac{1}{n_k}\left(\frac{dn}{d\theta}\right)_k\right]\right\}_k$$

$$= \left(\frac{dr_k}{d\theta}\right)_k\left[\left(\operatorname{cosec}^2\phi_k - 1\right)^{1/2} - \frac{1}{n_k}\left(\frac{dn}{d\theta}\right)_k\right] +$$

$$r_k \frac{d}{d\theta}\left[(\mathrm{cosec}^2\phi_k - 1)^{1/2} - \frac{1}{n_k}\left(\frac{dn}{d\theta}\right)_k\right]_k$$

$$= r_k\left[(\mathrm{cosec}^2\phi_k - 1)^{1/2} - \frac{1}{n_k}\left(\frac{dn}{d\theta}\right)_k\right]^2 + \tag{22}$$

$$r_k \frac{d(\mathrm{cosec}^2\phi_k - 1)^{1/2}}{d\theta} - r_k \frac{d}{d\theta}\left[\frac{1}{n_k}\left(\frac{dn}{d\theta}\right)_k\right]$$

The second term on the right hand side of (22) is:

$$r_k \frac{d}{d\theta}\left[\frac{1}{n_k}\left(\frac{dn}{d\theta}\right)_k\right]^2 = \frac{r_k}{2}\,\mathrm{cosec}^2\phi_k \tag{23}$$

the third term is:

$$r_k \frac{d}{d\theta}\left[\frac{1}{n_k}\left(\frac{dn}{d\theta}\right)_k\right] = \frac{r_k}{n_k}\left(\frac{d^2n}{d\theta^2}\right)_k - \frac{r_k}{n_k^2}\left[\left(\frac{dn}{d\theta}\right)_k\right]^2 \tag{24}$$

finally,

$$\left[\frac{d^2r}{d\theta^2}\right]_k = \frac{3}{2} r_k\,\mathrm{cosec}^2\phi_k - r_k - \frac{2r_k}{n_k}(\mathrm{cosec}^2\phi_k - 1)^{1/2}\left(\frac{dn}{d\theta}\right)_k \tag{25}$$

The equation for ray trajectory is

$$r_{k+1} = r_k + \varepsilon r_k\left[(\mathrm{cosec}^2\phi_k - 1)^{1/2} - \frac{1}{n_k}\left(\frac{dn}{d\theta}\right)_k\right] + \frac{\varepsilon^2}{2} r_k\left[\frac{3}{2} r_k\,\mathrm{cosec}^2\phi_k - 1 - r_k - \frac{2r_k}{n_k}(\mathrm{cosec}^2\phi_k - 1)^{1/2}\left(\frac{dn}{d\theta}\right)_k\right]. \tag{26}$$

For the two models [equations (19) and (26)] we consider:

a) $N = (n - 1) * 10^6$ (refractivity definition).

b) The variation of refractivity with height is assumed to be an exponential function:

$$N = N_o e^{-Q(r - r_o)}$$

c) To compute the refractivity gradient, (dN/dr) k, we can use two methods. First we use the Smith and Weintraub expression for each altitude; on the other hand, we can obtain by regression the refractivity curve and $(\Delta N/\Delta r)$ values (where $\Delta r = 100$ m).

d) To compute the ϕ_k angle, we use the geometry shown in figure 4.

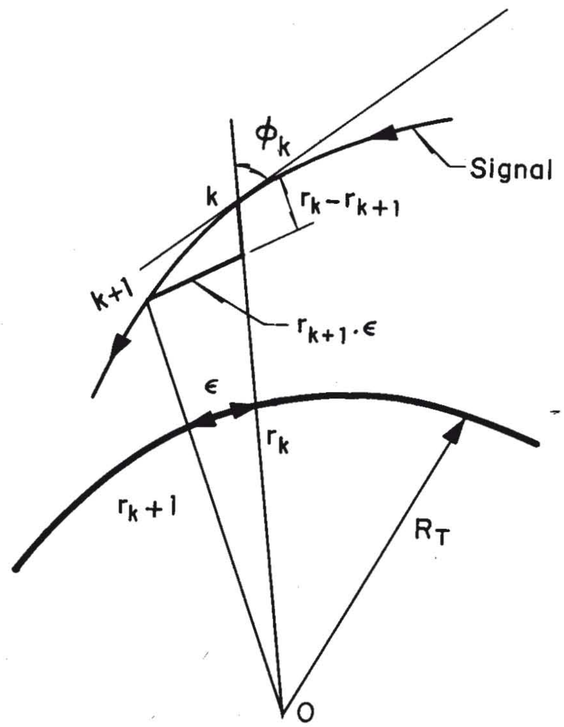

Fig. 4

where:

$$\mathrm{tg}\,\phi_k = \frac{\varepsilon\, r_{k+1}}{r_k - r_{k+1}}$$

and

$$\left(\frac{dn}{dr}\right)_k = \frac{n_k - n_{k+1}}{r_k - r_{k+1}}$$

RESULTS

For these cases we consider the station located in Mendoza (Argentina).

Data were taken with radiosondes in February 1, 1984 at 00.00 hs L.T.; and we also use mean data from twenty years ago. To process data we utilize only models I and II [equations (19 and (26)]. In both cases results agree very well with the path shape. In each case we use arriving angle equal to 26°, 46°, 66° and 86°, respectively.

For the unfavorable case ($\alpha = 26^o$), the horizontal length corresponding to the arrival of the signal in the troposphere is 17000 m and the vertical length for 26° and 31° only differs in 15 and 8 m, respectively. That is to say, it represents one part per million approximately. This assumes that the troposphere has horizontal stratification.

Note the strong influence in ray tracing of the change in water vapor pressure gradient at 2000 and 6000 m.

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