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Thermal feedback during viscous flow in cylindrical conduits in media with finite thermal properties

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RESUMEN

Se presenta un algoritmo por diferencias finitas explícitas para obtener soluciones aproximadas de la ecuación de balance de calor en una dimensión para un flujo con disipación viscosa. Se considera que la viscosidad es una función exponencial de la temperatura y que el fluído circula en un conducto cilíndrico en el que la pared posee características térmicas finitas y diferentes a cero. En trabajos previos, en los que sólo se ha considerado la impedancia interna del fluído, se encontró que para ciertos valores de las características físicas y geométricas del sistema, éste se torna inestable. En este artículo demostramos que las propiedades térmicas de la pared del sistema son de gran importancia para determinar el tiempo de ocurrencia de la inestabilidad y que ésta puede ocurrir incluso en los casos en que las paredes isotérmicas predicen estabilidad. Un mecanismo de este tipo se ha propuesto para explicar la ocurrencia de intrusiones magmáticas y la periodicidad de erupciones en volcanes poligenéticos.

PALABRAS CLAVE: Retoalimentación térmica, inestabilidades térmicas, generación viscosa de calor, flujo de magma en conductos.

ABSTRACT

An explicit finite-difference scheme to compute solutions of the heat equation with a generation term is presented. The heat is produced by viscous dissipation in a fluid where the viscosity is an exponential function of the temperature. Cylindrical conduits of infinite extent and walls of non-zero, finite thermal characteristics are considered. In earlier work, where only the internal impedance of the fluid was considered, thermal instabilities occurred for some combinations of the physical and geometrical parameters of the system. In this paper we show that the thermal properties of the boundary walls are of great importance to determine the time of occurrence of the instability and that instability can occur even in cases where the isothermal boundary case predicts stability. A mechanism of this type may account for the occurrence of magma intrusions and the periodicity of eruptions in polygenetic volcanism.

KEY WORDS: Thermal feedback, thermal instabilities, viscous generation of heat, magma flow in conduits.

INTRODUCTION

Gruntfest (1963) and Gruntfest *et al.* (1964) studied the thermal behavior of viscous fluids circulating in conduits of infinite length. As a result of viscous heat generation, even in the case of isothermal boundary conditions, thermal instabilities may ensue if the rate of heat generation is greater than a certain number now known as the Gruntfest number. Several researchers have examined the process in connection with the flow of magma in the asthenosphere or in volcanic conduits (e.g., Shaw, 1969; Fujii and Uyeda, 1974; De la Cruz-Reyna, 1976; Melosh, 1976; Spohn, 1980; Nelson, 1981). More recently Fujii (1983) has presented a model of polygenetic volcanism based on this process. Some of these authors have suggested that instabilities in flowing magmas may play a relevant role in the occurrence of lava fountains, magma intrusions and similar phenomena. It is certain that a temperature increase due to viscous dissipation must occur in flowing magmas. The resulting increase in temperature has important consequences for the dynamics of the magma and other mechanisms such as gas solubility. In earlier work, the walls of the conduits were assumed to be either isothermal or adiabatic: but the rock matrix is neither perfectly insulating nor removes heat instantaneously. In this paper we present an explicit finite-difference (F-D) scheme for computing the temperature profile and heat production in a cylindrical pipe with arbitrary values of the thermal conductivity and effective heat capacity of the wall. Solutions to this problem in rectangular coordinates (i.e. for flow in dyke-like conduits) have been given elsewhere (Espíndola and De la Cruz-Reyna, 1988).

THEORY AND FINITE-DIFFERENCE SCHEME

Consider a fluid circulating in a cylindrical conduit of radius R whose axis coincides with the z'axis. Let the temperature be $T' = T'(r', t')$ and the only non-zero component of velocity $V' = Vz'(r', z')$, where t' is the time and r' the radial coordinate. Consider also η, the viscosity, and K_1, C_1, K_2, C_2 the thermal conductivity and heat capacity (per unit volume) of fluid (medium 1) and matrix (medium 2) respectively (Figure 1).

The equation of balance of heat is given by

$$C_1 \frac{\partial T'}{\partial t'} = K_1 \left[\frac{\partial^2 T'}{\partial x'^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] + \sigma_{rz} \frac{\partial v'}{\partial r} \quad \text{in medium 1} \qquad (1)$$

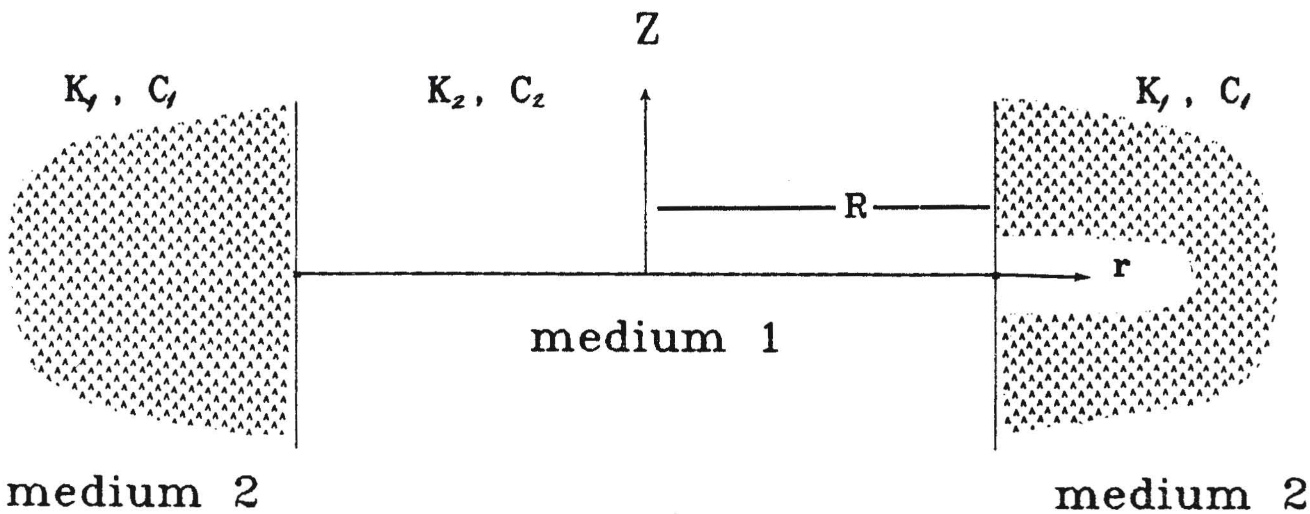

Fig. 1. Geometry of the problem. Medium 1 is a viscous fluid of thermal conductivity and diffusivity K_1 and C_1 respectively. Medium 2 is a rock matrix of thermal characteristics K_2 and C_2.

where σ_{rz} is the viscous stress.

If the fluid is Newtonian the constitutive relationship in the case being considered is given by:

$$\sigma = \sigma_{rz} = \eta \frac{\partial v'}{\partial r} \tag{2}$$

The viscosity of magmas may be approximated by:

$$\eta = \eta_o e^{-\alpha (T' - T_o)} \tag{3}$$

where T_o is a reference temperature and α is a constant that depends on the material. This approximation is valid only within a certain range of temperatures; but these cover the range of interest in the case of the flow of magmas (Shaw, 1969; Nelson, 1981). Thus, equation (1) becomes:

$$C_1 \frac{\partial T'}{\partial t'} = K_1 \left[\frac{\partial^2 T'}{\partial r'^2} + \frac{1}{r} \frac{\partial T}{\partial r'} \right] + \frac{\sigma^2}{\eta_o} e^{\alpha (T' - T_o)} \tag{4}$$

This equation is coupled with the equation of motion through σ. However, for steady flow in a pipe (Poiseuille flow), the equation may be uncoupled and solved independently. The appropriate expression of the Navier-Stokes equation is:

$$\eta \left[\frac{\partial^2 v'}{\partial r'^2} + \frac{1}{r'} \frac{\partial v'}{\partial r'} \right] = P + g\rho \tag{5}$$

where $P = \partial p / \partial z$ is the pressure gradient. Neglecting body forces and integrating (r' is an integrating factor) we find

$$\eta \frac{\partial V'}{\partial r'} = P r' \tag{6}$$

The left-hand side is the stress which can be substituted into equation (4) with the following choice of dimensionless variables:

$$t = t' K_1 / C_1 R^2$$

$$r = r' / R$$

$$T = \alpha (T' - T_o)$$

which yields

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + G r^2 e^T \tag{7}$$

where G, the Gruntfest number, is given by:

$$G = \frac{\alpha R^4 P^2}{4 K_1 \eta_o} \tag{8}$$

If the conduit is surrounded by a medium of thermal characteristics C_2 and K_2 (Figure 1), then the conditions at the boundary are as follows. Continuity of temperature:

$$\lim_{x' \to R^-} T'(r', t') = \lim_{x' \to R^+} T'(r', t') \tag{9}$$

Continuity of heat flux

$$K_1 \frac{\partial T'(R, t')}{\partial x'} = K_2 \frac{\partial T'(R, t')}{\partial x'} \tag{10}$$

The equation of heat in medium 2, using the same dimensionless variables, is:

$$\frac{\partial T}{\partial t} = D_{12} \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] \quad \text{in medium 2} \tag{11}$$

where:

$$D_{12} = K_{12}/C_{12} = \frac{K_1/K_2}{C_1/C_2} \qquad (12)$$

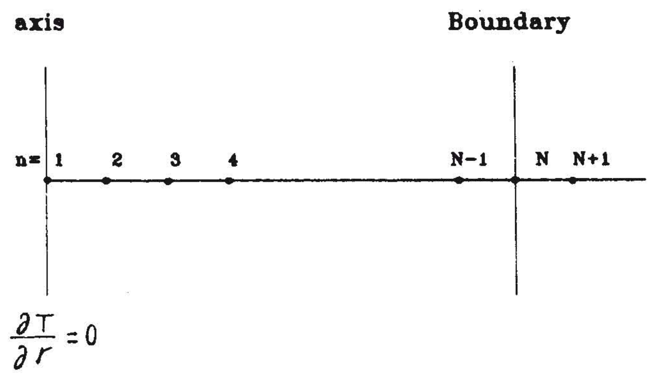

Fig. 2. Discretization mesh. Dr is the mesh aperture, and Dt is the time step.

By symmetry (Figure 2) one can solve the problem from the center of the conduit, requiring that

$$\frac{\partial T(0,T)}{\partial r} = 0 \qquad (13)$$

at this point. Equations (7) and (11) together with conditions (9), (10) and (13) can be approximated by finite-difference (F-D) equations, using a mesh of size Δr and time step Δt (Figure 2). The following F-D approximations are suitable to this purpose.

(i) At point n = 1 (r = 0),

$$T(1, m+1) = T(1, m) + 4\lambda\,[T(2, m) - T(1, m)] \qquad (14)$$

(see Smith, 1978).

(ii) At points n = 2 to N - 1 (equation 7)

$$T(n,m+1) = T(n,m) + \lambda\,[T(n-1,m) - 2T(n,m) + T(n+1,m)] + \frac{\lambda}{2(n-1)\Delta x}\,[T(n-1,m) - T(n+1,m)] + G\left[(n-1)\Delta x^2\, e^{T(n,m)}\,\Delta t)\right] \qquad (15)$$

(iii) At point n = N (the interface)

$$T(N,m+1) = T(N,m) + \frac{2\lambda}{1+C_{12}}\,[T(N-1,m) - (1+K_{12})\,T(N,m) + K_{12}\,T(N+1,m)] + \frac{Ge^{T(n,m)}\Delta t}{1+C_{12}}, \qquad (16)$$

where $\lambda = \Delta t/\Delta x^2$. We have written the temperature at the n-th node and m-th time step as T(n, m) (see for instance Carnahan *et al*, 1969).

Note that in both equations we use the temperature of the previous time step in the exponential term. We have found that a Taylor's series estimate of T at time m + 1/2, as described for instance in Von Rosenberg (1969, p. 57), improves the accuracy. However, the same effect is achieved by simply decreasing the time-step, generally a less time-consuming procedure.

Note that formula (16) for $K_{12} = C_{12} = 0$ yields the same expression that would be obtained if the derivative of T were zero at n = N (i.e. the adiabatic case). It also yields

$$\lim_{\substack{K_{12} \to \infty \\ X_{12} \to \infty}} T(N, m+1) = T(N, m) \quad , \qquad (17)$$

that is, the isothermal case.

Formula (16) requires a node at point N + 1. In order to supply this node, one is required to continue the computation of temperature by a F-D approximation to equation (11). This would have to be carried out for a number of points and an artificial boundary be placed at a distance such that its presence would not introduce spurious "reflections". The large number of time steps required in some calculations could make this distance excessive. However, since there are analytical solutions to the problem of a whole space subjected to temperature changes at the surface of a cylindrical source, one may treat the changes in temperature at node N as step changes in a cylindrical source buried in medium 2. This approach provides the following formula to compute the temperature at the required point:

$$T(N+1,K+1) = \sum_{k=1}^{K} \Delta T(k) \left[\left(\frac{1}{r^{1/2}} - \frac{\Delta r^2}{4r^{3/2}}\right) \mathrm{erfc}\, \frac{\Delta x}{2[D_{12}\Delta t\,(K-K+1)]^{1/2}} + \frac{\Delta x\,(D_{12}\Delta t\,K)^{1/2}}{r^{3/2}} \exp(-\Delta r^2/4D_{12}\Delta t\,K)\right] \qquad (18)$$

where the first two terms are from the series solution given by Carslaw and Jaeger (1959, p. 336; see also p. 484), and $\Delta T(k)$ is the change in temperature at point N at time $t = K\,\Delta t$ and $r = 1 + \Delta r$. Therefore, the scheme requires solutions of equations (14), (15), (16), and (18). The numerical stability of the system is the same as for the simple heat conduction equation, i.e.:

$$D_{12}\ \Delta t/\Delta x^2 \leq 2/5 \quad . \tag{19}$$

We may also compute, following Fujii and Uyeda (1974), the heat production as a function of time. In the constant stress case the highest temperature occurs in the center of the conduit. This is not the case in Poiseuille flow, and a plot of heat production is a better description of the behavior of the system. The dimensional dissipative heat production per unit length along the cylindrical conduit is computed from:

$$Q = 2\pi \int_0^R \sigma u r' dr' \quad , \tag{20}$$

where σ and u are the stress and the velocity gradient.

For Poiseuille flow the local stress is given by:

$$\sigma = -\frac{P}{2} r \quad , \tag{21}$$

and the radial velocity gradient by:

$$u = \frac{\partial v}{\partial r} = \frac{P}{2\eta}\frac{R}{r} \quad . \tag{22}$$

Hence

$$Q = \frac{\pi P^2}{2} \int_0^R \frac{r^9}{\eta_o} e^{a(T-T_o)} dr \quad . \tag{23}$$

At time t = 0 we have

$$Q_o = \frac{\pi p^2}{8\eta_o R^4} \tag{24}$$

Finally, the dimensionless heat production as referred to the initial time is, normalizing with respect to Q_o:

$$Q/Q_o = 4 \int_0^1 r^3 e^T dr \quad . \tag{25}$$

This integral can be easily evaluated by linear interpolation.

RESULTS AND DISCUSSION

From the features of the model, it is evident that the behavior of the system depends not only on the value of G for the adiabatic and isothermal cases, but also on the contrast in conductivity and heat capacity between fluid and matrix in the general case. Some typical values of the relevant properties of magmas as given by Murase and McBirney (1973) are:

Viscosity	η= 10 to 10^7 poises
Thermal conductivity	k = 10^5 to 4.2 x 10^5 ergs/cm sec°C
Mass density	ρ= 2.2 to 2.8 gr/cm^3
Rheological constant	α = 0.015°C^{-1}

With these values, G and K_{12} range between 9 x 10^{-16} $R^4 P^2$ and 3.8 x 10^{-9} $R^4 P^2$, and 0.25 to 4 respectively. If the pressure gradient arises from lithostatic forces, then P = g Δρ where g is the acceleration of gravity and Δρ the density contrast. Thus, typical values of G range between 0.54 and 420 for conduits of 1 meter of radius, though higher values may be attained for wider conduits due to the fourth power of this parameter in the value of G. As stated by Jaeger (1964), the estimation of the heat capacity in partially molten rocks becomes cumbersome due to the latent heat of melting; one way to account for it is including it in an "effective" heat capacity computed from:

$$C^* = \rho\ (Cp + Ql/\Delta T) \tag{26}$$

where ΔT is the melting range of the magma, Ql is the latent heat of melting and Cp is the mean heat capacity. Shaw (1969) gives a value of C* for basalts of about 10^8 erg/cm^3 °C. Smislov *et al.* (1979) also published plots of the heat capacity as a function of temperature for igneous rocks of different compositions. The ratio of the higher to the lower value for their samples is less than 2, with values that range from around 2.5 to 4.5 x 10^7 erg/cm^3 °K. It is clear that the variations in G can be much larger than those in conductivity and heat capacity.

The system is less sensitive to changes in heat capacity contrast than to changes in the conductivity contrast. In fact, the effect of both variables is combined in the diffusivity contrast between both materials, except in equation (16) where the heat capacity appears alone.

The following results illustrate the behavior of the system when the effect of the thermal properties are considered as compared to the adiabatic and isothermal cases. Figure 3 shows values of the temperature as a function of dimensionless time at different distances from the axis, for the isothermal case when G = 0.2. The results agree with those obtained by Gruntfest *et al.* (1964) with an analog computer. Figure 4 shows plots of the normalized heat production against time for three values of G and several values of the contrast in conductivity. Note that for G = 1 the system is stable in the isothermal case; however, for finite contrasts in conductivity, the temperatures rise monotonously at larger values of K_{12} and become unstable for small values. Thus, in the general case the system never reaches a steady state even though for small values of G and K_{12} heat is produced at rates that permit a slow warming of the whole system (magma and matrix rock) without a runaway situation being generated. However, this happens only for very small values of G. At G = 5, which should be stable in the isothermal case, the behavior even at larger values of K12 is more like the adiabatic case than like the isothermal. For G = 10, where even the isothermal case predicts instability, the general case shows instability at shorter times (and the results are not significantly different for other values of C12).

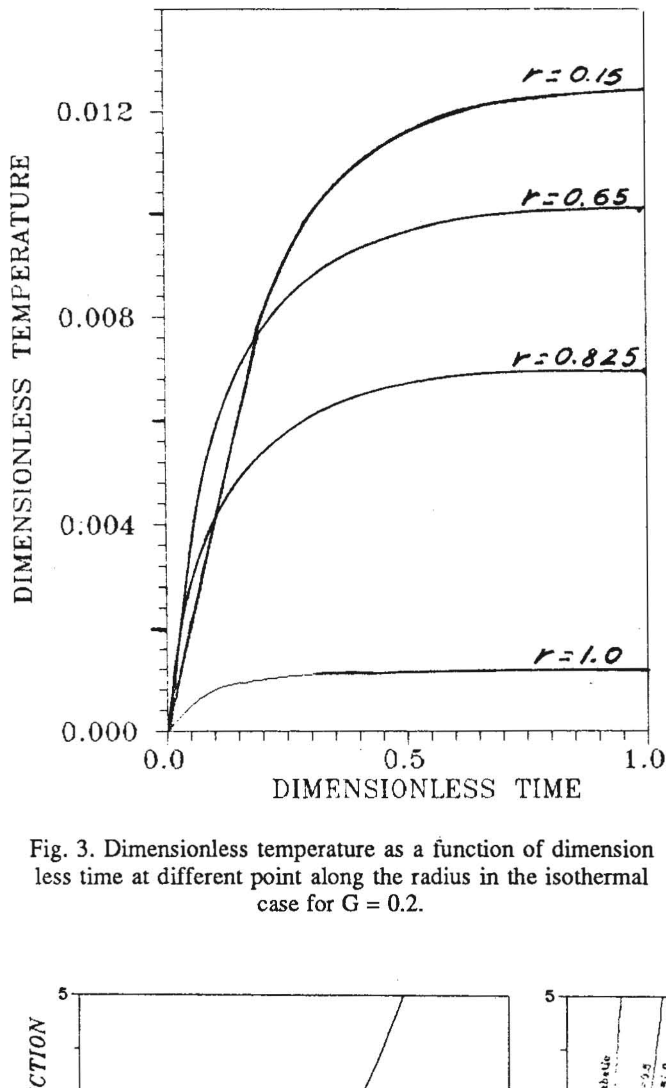

Fig. 3. Dimensionless temperature as a function of dimension less time at different point along the radius in the isothermal case for G = 0.2.

Figure 5 shows the normalized heat production as a function of time for different values of G. The adiabatic case is always unstable. A change in G only affects the time to reach instability. In the isothermal case the critical value of G is about 8 as reported by Gruntfest *et al.* (1964). In the general case (using D12 = 1, though different values do not modify significantly our conclusions), and for values of G as small as 3, the flow becomes unstable. Figure 6 displays, as an example, the temperature profiles for G = 11 in all three cases. The maximum temperature occurs at the boundary in the adiabatic case, but not in the other two cases. The general case resembles a weighted average of the adiabatic and isothermal cases.

In Figure 7 we have plotted the time to reach instability for the three types of boundary. Note the asymptotic behavior of the curves: the asymptote is Gc in the isothermal case and 0 in the adiabatic one. The general case (with D12 = 1) lies between both curves (though closer to the adiabatic one) and tends to be asymptotic to a value of around 2. The smaller the value of D12 the closer the curve is shifted to the left. For very large values of G all curves tend to the same value, since (for large rates of heat generation) the finite conductivity of the fluid itself becomes the main factor which impedes heat transfer towards the boundaries.

CONCLUSIONS

We have shown that the thermal properties of the walls during Poiseuille flow greatly affect the thermal evolution. Estimates of temperatures, and models based on simplified calculations assuming isothermal or adiabatic conditions, should be revised. For small Gruntfest numbers, the adiabatic model overestimates the temperatures while the isothermal model underestimates them. However, the general case is closer to the adiabatic than to the isothermal

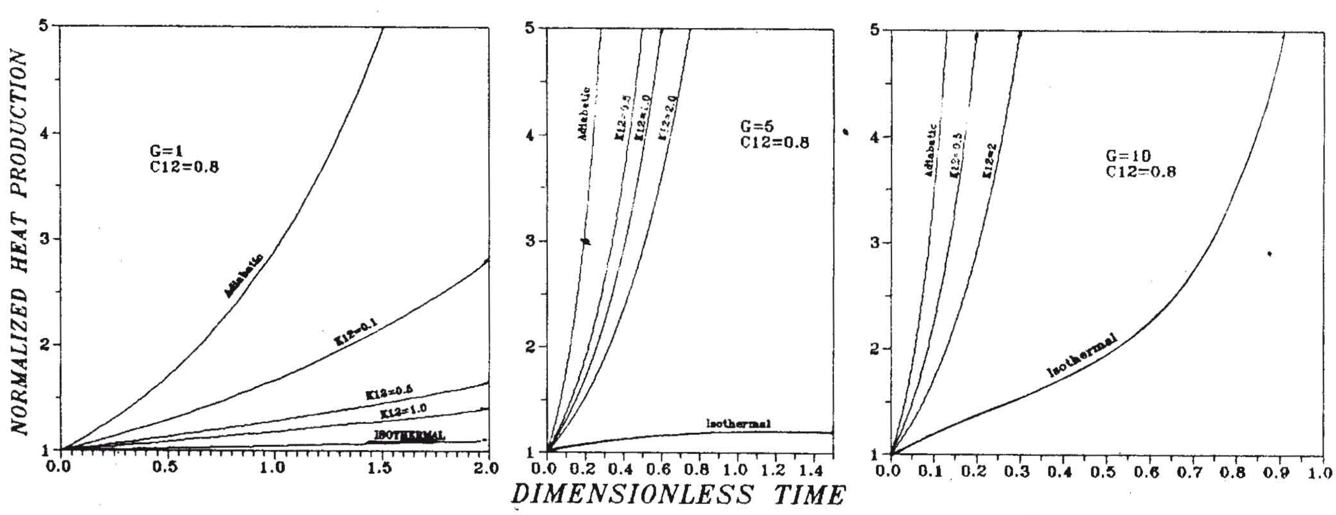

Fig. 4. Normalized heat production as a function of dimensionless time for Poiseuille flow and several values of K12 and G.

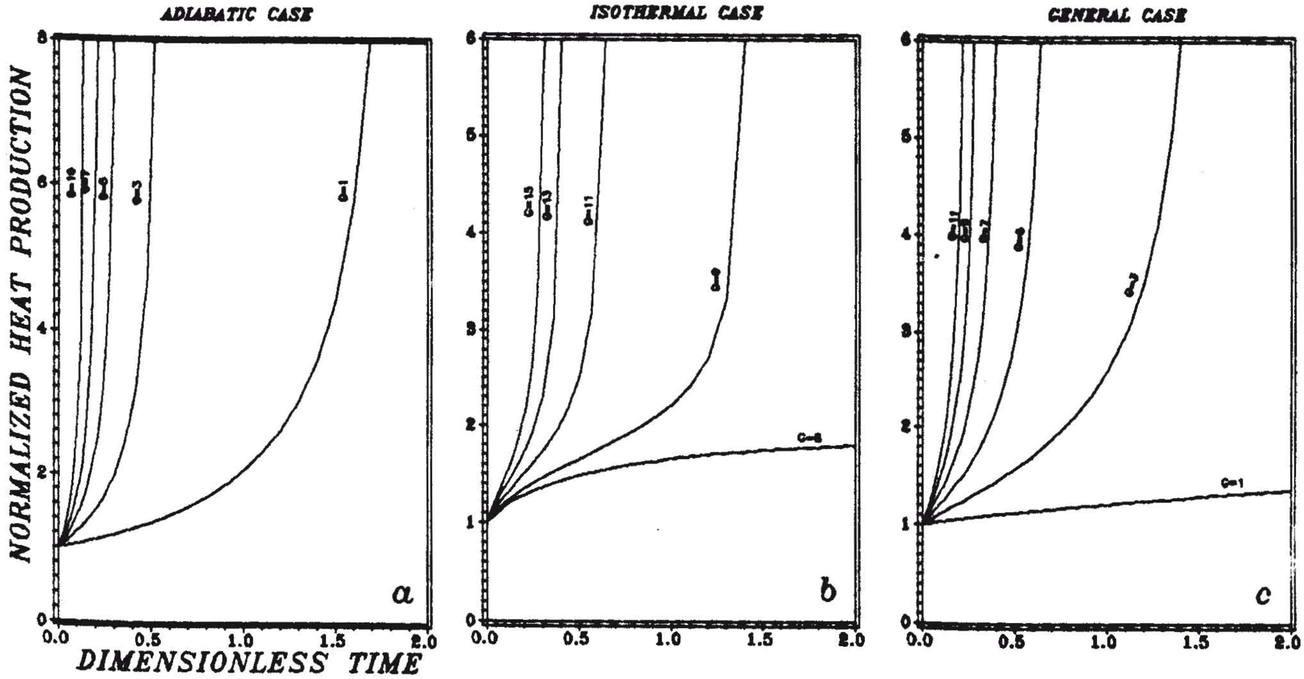

Fig. 5. Normalized heat production as a function of dimensionless time for: (a) adiabatic case; (b) isothermal case, and (c) general case ($D_{12} = 1$)

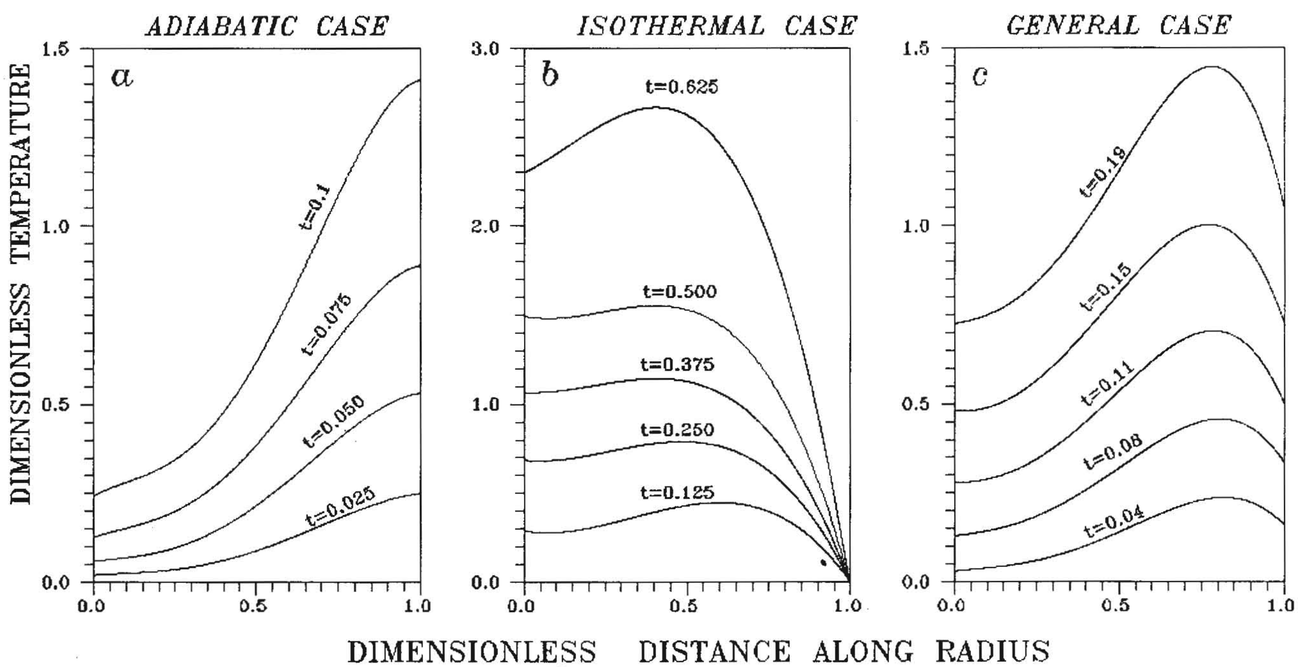

Fig. 6. Temperature profiles at G = 11 for: (a) adiabatic case; (b) isothermal case, and (c) general case ($D_{12} = 1$).

model for reasonable values of the contrast of diffusivity between the media. All solutions tend to the same value for large values of G. In these cases the type of boundary is of less relevance than the internal thermal impedance of the fluid.

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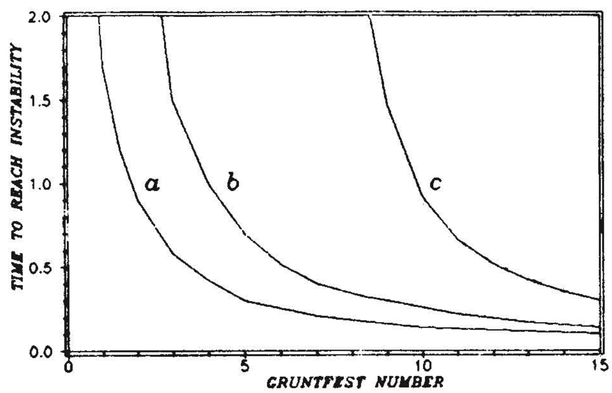

Fig. 7. Plot of the dimensionless time required to reach instability as a function of the Gruntfest number. Gc is the critical value for the isothermal case.

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