GEOFISICA INTERNACIONAL

COMMUNICATION

AN EXERCISE ON DIFFUSION BY CONTINUOUS MOVEMENTS

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RESUMEN

Se proponen expresiones analíticas que se comprenden mejor con los valores experimentales evaluados por Paul Frenzen, en 1963, para la Función Lagrangiana de Autocorrelación R (ξ) acerca de campos turbulentos generales en fluidos de diversas estabilidades gravitacionales. Se demuestra que las expresiones analíticas aquí propuestas, representan correspondencias en los datos experimentales, comparadas con las propuestas por Frenzen. Se incluye una descripción referente a la curva de correspondencia para cada grupo de valores experimentales considerados. Se ensayan soluciones generales para la ecuación de difusión de Taylor, empleando las expresiones analíticas propuestas para la Función Lagrangiana de Autocorrelación.

ABSTRACT

Analytical expressions are proposed and best fitted to the experimental values evaluated by Paul Frenzen in 1963, for the Lagrangian Autocorrelation Function R (ξ) of turbulent fields generated in fluids of several gravitational stabilities. It is demonstrated that the analytical expressions proposed herein represent better fits to the experimental data when compared with those proposed by Frenzen. A description is given concerning the best fit curve for each of the sets of experimental values considered. General solutions for Taylors diffusion equation are sought using the proposed analytical expressions for the Lagrangian Autocorrelation Function.

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INTRODUCTION

The basic foundations of the general theory of turbulence that applies to the concept of "Diffusion by Continuous Movements", was presented by G. I. Taylor in 1921. He was able to extend the concept of the description of Brownian motion known as "the Random Walk" to conditions more representative of fluid motion on the macroscale by providing for "continuous movements", i. e., the magnitude and probability of occurrence of the particles displacements were allowed to vary continuously in a manner prescribed by the character of the turbulent field.

Taylor's diffusion equation is given by:

$$[X^2] = 2\,[U^2] \int_0^T \int_0^t R\,(\xi)\, d\xi\, dt, \tag{1}$$

where:

$[X^2]$ = mean square distance in the X-direction
$[U^2]$ = mean square velocity in the X-direction
$R\,(\xi)$ = autocorrelation function
T = Time

The autocorrelation function $R\,(\xi)$, as defined by:

$$R\,(\xi) = \frac{[U_t\, U_{t+\xi}]}{[U^2]}, \tag{2}$$

expresses the correlation of the velocity of a particle at an initial time to the velocity of the same particle at a later time. For short periods of time, the velocities of individual different particles will tend to remain highly correlated, i.e.,

$$\lim_{\xi \to 0} R\,(\xi) = 1$$

Conversely, as ξ becomes sufficiently large, R (ξ) falls to zero, i. e.,

$$\lim_{\xi \to \infty} R(\xi) = 0$$

Nevertheless, since we are considering random motions of a limited number of particles, we expect that a significant fraction of these will repeat earlier velocities ocassionally. This being the case, our estimates of R (ξ) based on limited samples drawn from a larger population cannot be expected to maintain a value of exactly zero for all large ξ but instead they will be observed to oscillate around zero as a mean.

From the arguments set forth in the past paragraphs one is able to estimate *a priori* that the Lagrangian autocorrelation function will, in general, behave in the manner sketched in figure 1, at least during the initial and terminal periods.

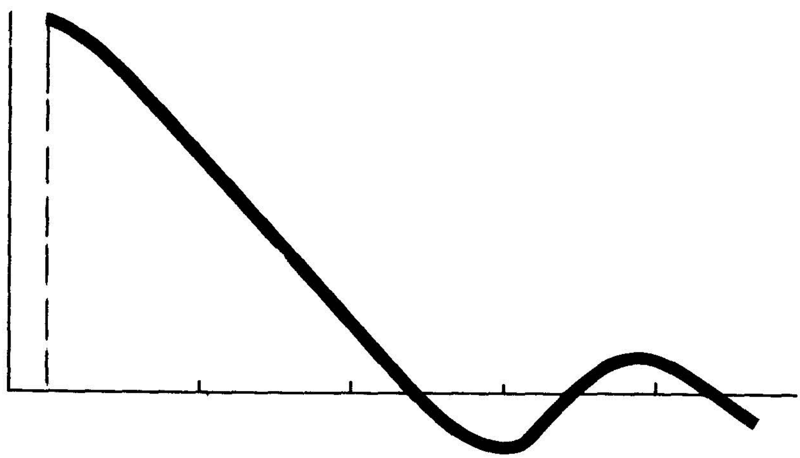

Figure 1. The nature of the Lagrangian Autocorrelation Function; R (ξ) $\simeq$ 1 for $\xi < \xi_1$; R (ξ) = 0 for $\xi > \xi_2$.

Frenzen (1963), determined that the behavior of the autocorrelation function for intermediate periods, i. e., when $\xi_1 < \xi < \xi_2$ in figure 1, cannot be anticipated, since in this range the rate at which R (ξ) decreases depends upon the specific properties of the particular field of turbulence for which the function was evaluated. He concluded that one of the factors which can influence the behavior of R (ξ) in this intermediate range is the gravitational stability of the fluid medium.

In solving Taylor's diffusion equation for two limiting cases, i. e., as T tends to zero, and as T tends to infinity, one can demonstrate the control exerted by the autocorrelation function upon the diffusion process.

Considering the case in which the time T tends to zero, we observe that X tends to zero, and consequently R (ξ) tends to unity. Therefore the inner integral of equation (1) becomes:

$$\int_0^t R\,(\xi)\,d\xi = \int_0^t 1\,d\xi = t \tag{3}$$

and the solution of the entire expression reduces to

$$[X^2] = 2\,[U^2]\int_0^T t\,dt = [U^2]\,T^2 \tag{4}$$

Therefore,

$$\sqrt{[X^2]} = T\,\sqrt{[U^2]} \tag{5}$$

or

$$\sqrt{[X^2]} \propto T \quad \text{when } T < \xi_1 \text{ (see fig. 1).} \tag{6}$$

Equation (6) indicates that the dispersing particles will initially occupy a conical plume, since the envelope extends linearly with time.

If we now consider the second limiting case, namely when T tends to infinity, we observe that X tends to infinity and consequently the inner integral of equation (1) attains a constant value I equal to the area under the R (ξ) curve, i. e.

$$\lim_{t \to \infty} \int_0^t R(\xi)\, d\xi = I \qquad (7)$$

Consequently R (ξ) = 0 for $T > \xi_2$. Therefore the solution of the entire expression will reduce to:

$$[X^2] = 2\,[U^2] \int_0^T I\, dt \qquad (8)$$

Therefore,

$$[X^2] = 2\,[U^2]\, I\, T \qquad (9)$$

$$\sqrt{[X^2]} = \sqrt{2\,[U^2]\, I\, T} \qquad (10)$$

or

$$\sqrt{[X^2]} \propto \sqrt{T}, \text{ when } T > \xi_2. \text{ (See figure 1).} \qquad (11)$$

Equation (11) indicates that the volume occupied by the dispersing particles eventually assumes the shape of a paraboloid.

In nature, these two envelopes of diffusion can be seen to succeed one another under atmospheric conditions in the initial and final *average* outlines of smoke plumes issuing from stacks.

FRENZEN'S EXPERIMENTS

In 1963, Paul Frenzen of the University of Chicago, performed a series of experiments in which the Lagrangian autocorrelation function was evaluated by observing the motions of particles in turbulent fields generated in fluids of several gravitational stabilities. The homogeneous turbulent circulation was generated by the passage of a rectangular grid of bars placed normal to the walls of a rectangular channel. The turbulence fields created by the passage of the rectangular grid underwent rapid decay, since the model contained no continuing source of turbulent energy. This limitation was overcame by applying Batchelor's decay corrections to the observed turbulent-velocity fluctuations.

The results of his experiments led him to conclude, among other things, that it was feasible to employ decay corrected, homogeneous fields of turbulence produced in the lee of grids as a medium for the study of turbulent diffusion, and that for this purpose Batchelor's decay correction procedure proved satisfactory. He was also able to observe that when stability is sufficiently strong the vertical component of the autocorrelation function assumed analytical expressions of the type,

$$R(\xi) = e^{-(\xi/a)^2} \tag{12}$$

ALTERNATIVE ANALYTICAL EXPRESSIONS FOR R (ξ)

The characteristic oscillatory behavior of R (ξ) as obtained by Frenzen, suggested that analytical expressions of the type

$$R(\xi) = e^{-a\xi} \operatorname{Cos} b\xi \tag{13}$$

and

$$R(\xi) = e^{-a\xi^2} \operatorname{Cos} b\xi_2 \tag{14}$$

would better fit the experimental values in question, as compared to the fit obtained by using Frenzen's analytical expression for R (ξ) as defined in equation (12).

Equation (13) allows for oscillations of continuous decreasing amplitudes at constant frequency, while equation (14) oscillates with continuous decreasing amplitudes at continuous increasing frequencies.

In subsequent pages a description will be given concerning the best fit curve for each of the sets of experimental data considered.

General solutions of Taylor's diffusion equation were sought by substituting the autocorrelation functions as defined in equation (13) and (14), and performing the integrations. At this point it is worthwhile mentioning that a solution of Taylor's diffusion equation using equation (14) as the analytical expression for the autocorrelation function, could not be determined, due to the fact that the integrals could not be evaluated analytically. This being the case, we only consider the subtitution of equation (13) in equation (1) and thus obtain:

$$[X^2] = 2\,[U^2] \int_0^T \int_0^t e^{-\xi/A} \operatorname{Cos} \xi/B \, d\xi \, dt \qquad (15)$$

where $a = 1/A$ and $b = 1/B$

Evaluating the inner integral in equation (15) yields:

$$\frac{[X^2]}{2\,[U^2]} = \int_0^T \frac{e^{-\xi/A}}{\left(\frac{1}{A}\right)^2 + \left(\frac{1}{B}\right)^2} \left\{ -\frac{1}{A} \operatorname{Cos} \xi/B + \frac{1}{B} \operatorname{Sin} \frac{\xi}{B} \right\}_0^t dt = \qquad (16)$$

$$\int_0^T \frac{A^2 B^2}{A^2 + B^2} \left\{ \frac{1}{A} - \frac{1}{A} e^{-t/A} \operatorname{Cos} \frac{t}{B} + \frac{1}{B} e^{-t/A} \operatorname{Sin} \frac{t}{B} \right\} dt$$

Integrating (16) yields:

$$\frac{[X^2]}{2\,[U^2]} = \frac{A^2 B^2}{A^2 + B^2} \left\{ \frac{1}{A} \int_0^T (1 - e^{-t/A} \operatorname{Cos} \frac{t}{B})\, dt + \frac{1}{B} \int_0^T e^{-t/A} \operatorname{Sin} \frac{t}{B}\, dt \right\}$$

$$\frac{A^2 B^2}{A^2 + B^2} \left\{ \frac{A^2 - B^2}{A^2 + B^2} + \frac{T}{A} \right\} + \left\{ \frac{(A^2 B^2)^2}{A^2 + B^2} e^{-T/A} \right\}$$

$$\left\{ \frac{B^2 - A^2}{A^2 B^2} \operatorname{Cos} \frac{T}{B} - \frac{2}{AB} \operatorname{Sin} \frac{T}{B} \right\} \qquad (17)$$

We now proceed to analyze how $\frac{[X^2]}{2[U^2]}$, behaves for the special cases when

$T >> 1$ and $T << 1$. Considering first the case in which $T << 1$, we subtitute the aproximations $\mathrm{Sin}\,\frac{T}{B} \simeq \frac{T}{B}$, $\mathrm{Cos}\,\frac{T}{B} \simeq \frac{1-T^2}{2B^2}$, and $e^{-T/A} \simeq \frac{1-T^2}{2A^2}$

in equation (17), since T is small. Consequently:

$$\frac{[X^2]}{2[U^2]} = \frac{A^2B^2}{A^2+B^2}\left\{\frac{A^2-B^2}{A^2+B^2} + \frac{T}{A}\right\} + \left\{\frac{A^2B^2}{A^2+B^2}\right\}\left\{\frac{1-T^2}{2A^2}\right\}^2$$

$$\left\{\frac{B^2-A^2}{A^2B^2}\left(\frac{1-T^2}{2B^2}\right) - \left(\frac{2T}{AB^2}\right)\right\} = \frac{1}{A^2+B^2}\left\{\frac{B^2-A^2}{A^2+B^2} + AT\right\} + \left\{\frac{1}{A^2+B^2}\right\}^2$$

$$\left\{1 - AT + A^2\frac{T^2}{2}\right\}\left\{(A^2-B^2)\left(1-\frac{B^2T^2}{2}\right) - (2AB)(BT)\right\} =$$

$$\frac{T^2(A^2+B^2)^2}{2(A^2+B^2)^2} = \frac{T^2}{2} \tag{18}$$

Therefore, when T is much smaller than unity, the value of:

$$\sqrt{\frac{[X^2]}{[U^2]}} = T$$

In the case where T >> 1, we substitute the aproximation $e^{-T/A} \simeq 0$, in equation (17), since T is large. Consequently:

$$\frac{[X^2]}{2[U^2]} = \left\{ \left(\frac{A^2 B^2}{A^2 + B^2} \right) \left(\frac{A^2 - B^2}{A^2 + B^2} + \frac{T}{A} \right) \right\} =$$

$$\left\{ \frac{A^4 B^2 - A^2 B^4}{(A^2 + B^2)^2} + \frac{AB^2 T}{A^2 + B^2} \right\} \qquad (19)$$

If we compare the constant term $\frac{A^4 B^2 - A^2 B^4}{(A^2 + B^2)^2}$, with the large value of T, we see that it is very small, and that it can be neglected. This being the case, equation (19) can be reduced to

$$\frac{[X^2]}{2[U^2]} = \frac{AB^2 T}{A^2 + B^2} \qquad (20)$$

METHODOLOGY FOR FITTING THE ALTERNATIVE ANALYTICAL EXPRESSIONS OF R (ξ) TO FRENZEN'S EXPERIMENTAL VALUES

The alternative analytical expressions for R (ξ) were best fitted to a set of Frenzen's observed experimental values of R (ξ) by the least mean square error technique. The chosen set of observed experimental values of R (ξ) are those presented in figures two and three of this report, and correspond to those values obtained by Frenzen (1963), and which he reported in his doctoral dissertation in figures 24 and 34 of pages 90 and 94 respectively.

The procedure to fit the alternative analytical expressions was to let the correlation function be $R(\xi)$ with parameters a, b. The error between the experimental values and $R(\xi)$ was calculated by setting

$$F(a, b) = \sum_{i=1}^{N} (R(\xi_i) - R_i)^2 \tag{21}$$

where F (a, b) represents a function which defines the mean square error, and R_i are the experimental values corresponding to ξ_i. Subsequently, the function F (a, b) was minimized. The minimum value of F, corresponds to a and b such that $\frac{\partial F}{\partial a} = 0$ and $\frac{\partial F}{\partial b} = 0$. Finally the partial derivatives of F with respect to a and b were solved to determine the value of a and b.

The two autocorrelation functions as defined in equation (13) and (14) were fitted to each set of observed experimental values thus yielding four F type functions. For the set of observed experimental values plotted in figure 2 of this report we obtain:

$$F(a. b) = \sum_{i=1}^{N} \left\{ e^{-a\xi_i} \operatorname{Cos} b\xi_i - R(i) \right\}^2 \tag{22}$$

and

$$F(a, b) = \sum_{i=1}^{N} \left\{ e^{-a\xi_i^2} \operatorname{Cos} \xi_i^2 - R(i) \right\}^2 \tag{23}$$

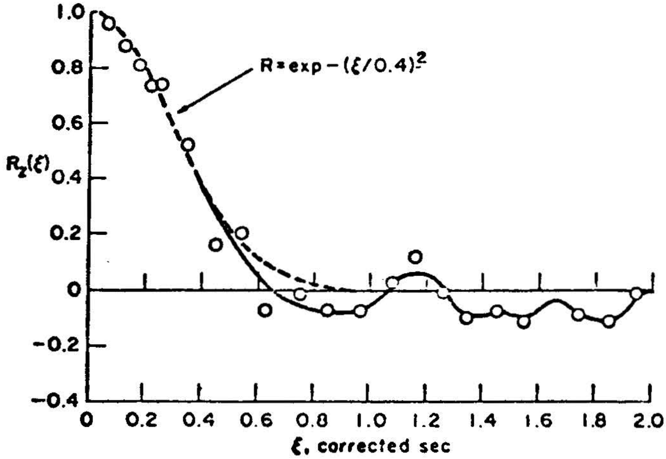

Figure 2. Frenzen's figure 24 illustrating a correlogram for his experiment s 25/z. The alternative analytical expressions for R (ξ) were best fitted to this set of observed experimental values of R (ξ). (From Frenzen, 1963).

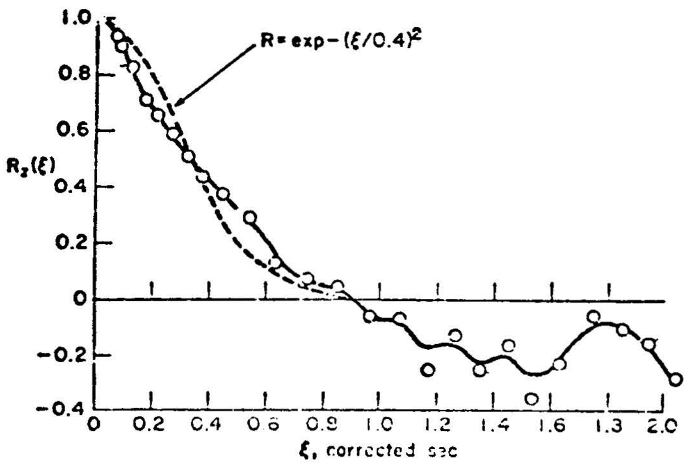

Figure 3. Frenzen's figure 32. illustrating a correlogram for his experiment s 40/z. The alternative analytical expressions for R (ξ) were best fitted to this set of observed experimental values of R (ξ). (From Frenzen, 1963).

For the observed experimental values plotted in figure 3 of this report we obtain:

$$F(a, b) = \sum_{i=1}^{N} \left\{ e^{-a\xi_i} \mathrm{Cos}\, b\xi_i - R(i) \right\}^2 \tag{24}$$

and

$$F(a, b) = \sum_{i=1}^{N} \left\{ e^{-a\xi_i^2} \mathrm{Cos}\, b\xi_i^2 - R(i) \right\}^2 \tag{25}$$

The final best fit expressions for both set of observed experimental values have been plotted in the form of correlograms and are presented in figures 4 and 5, respectively.

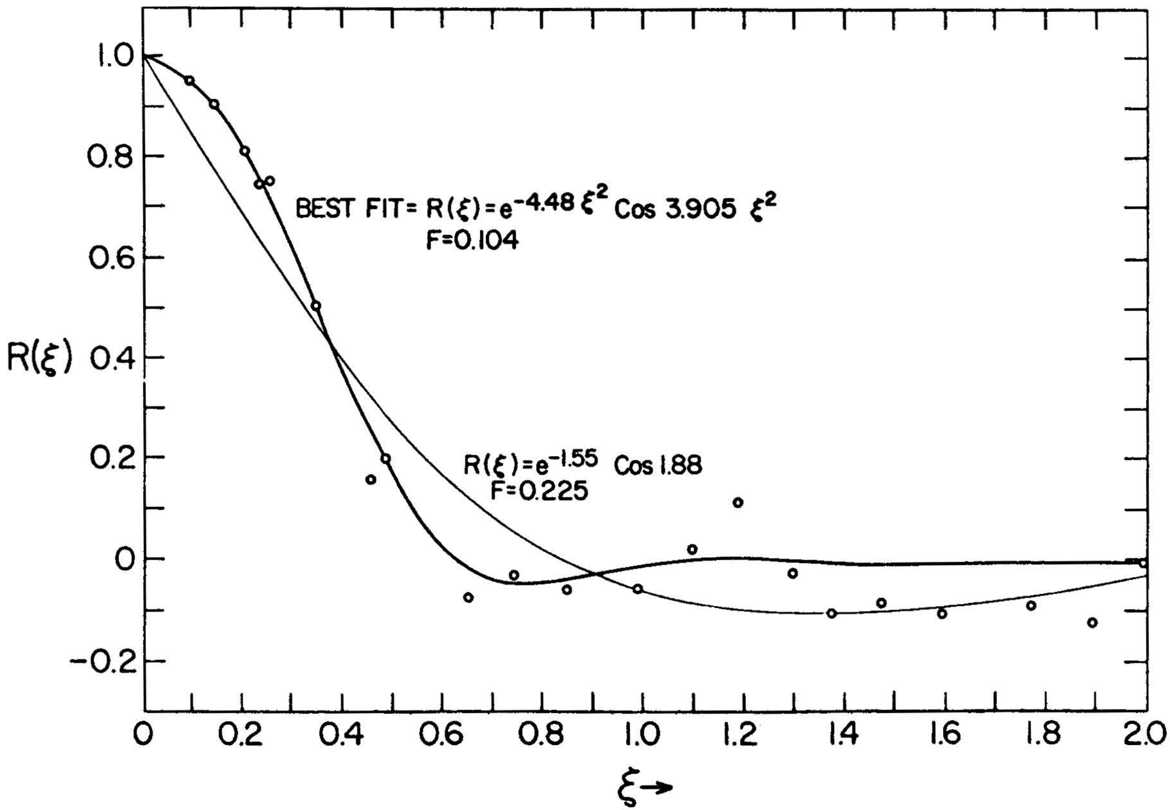

FIGURE 4

CORRELOGRAM USING FRENZEN'S EXPERIMENTAL DATA S/25/Z

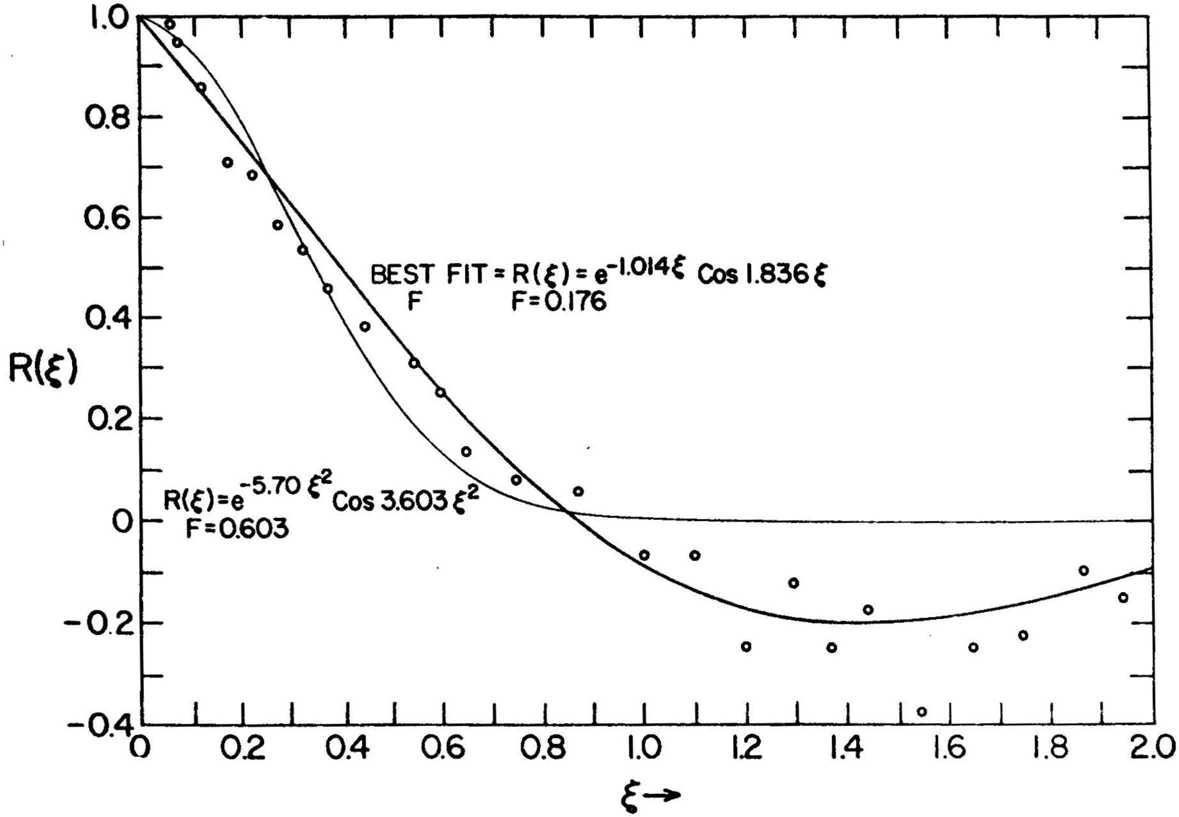

FIGURE 5

CORRELOGRAM USING FRENZEN'S EXPERIMENTAL VALUES S/40/Z

GRAPHICAL REPRESENTATION OF $\frac{[X^2]}{2[U^2]}$ AS A FUNCTION OF TIME

Figures 6, 7, 8 and 9 illustrate the behavior of the ratio of the mean square displacements to the mean square velocities as a function of small and large values of time for the set of observed experimental values shown in figures 2 and 3, respectively.

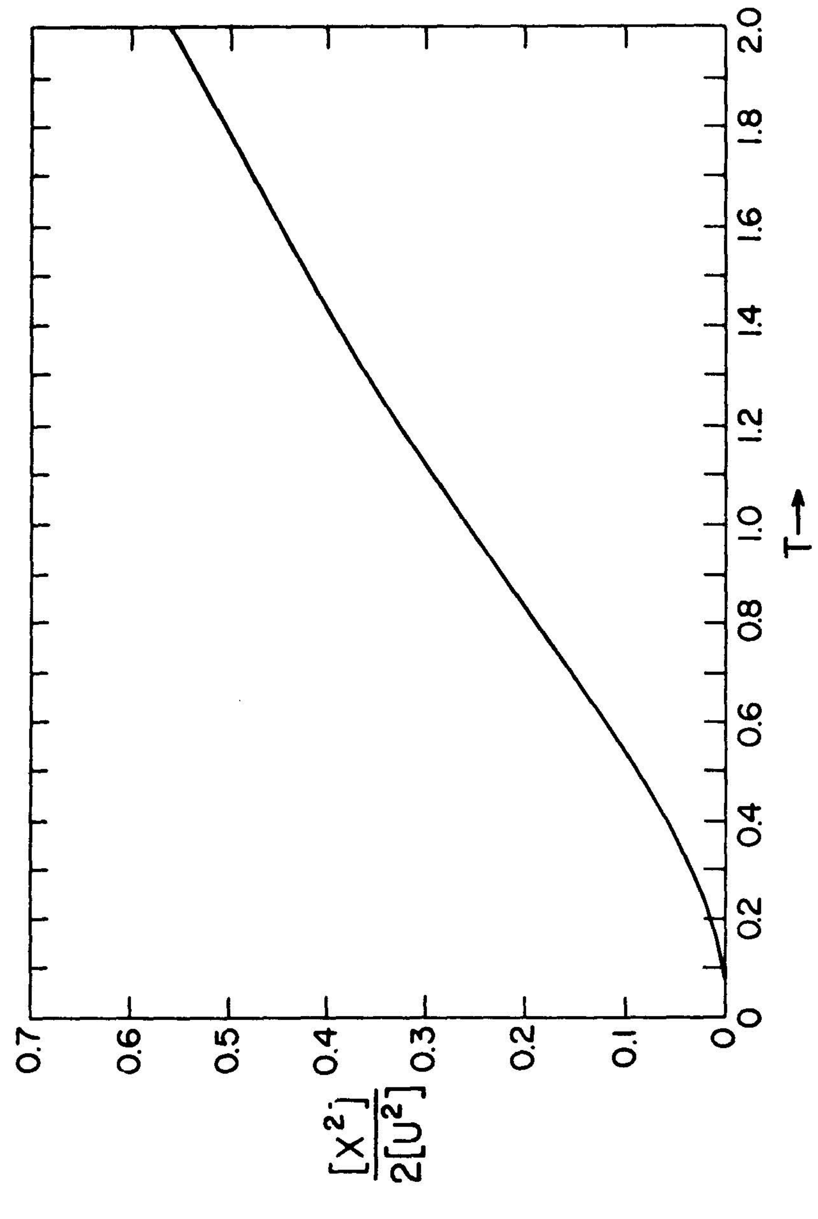

FIGURE 6

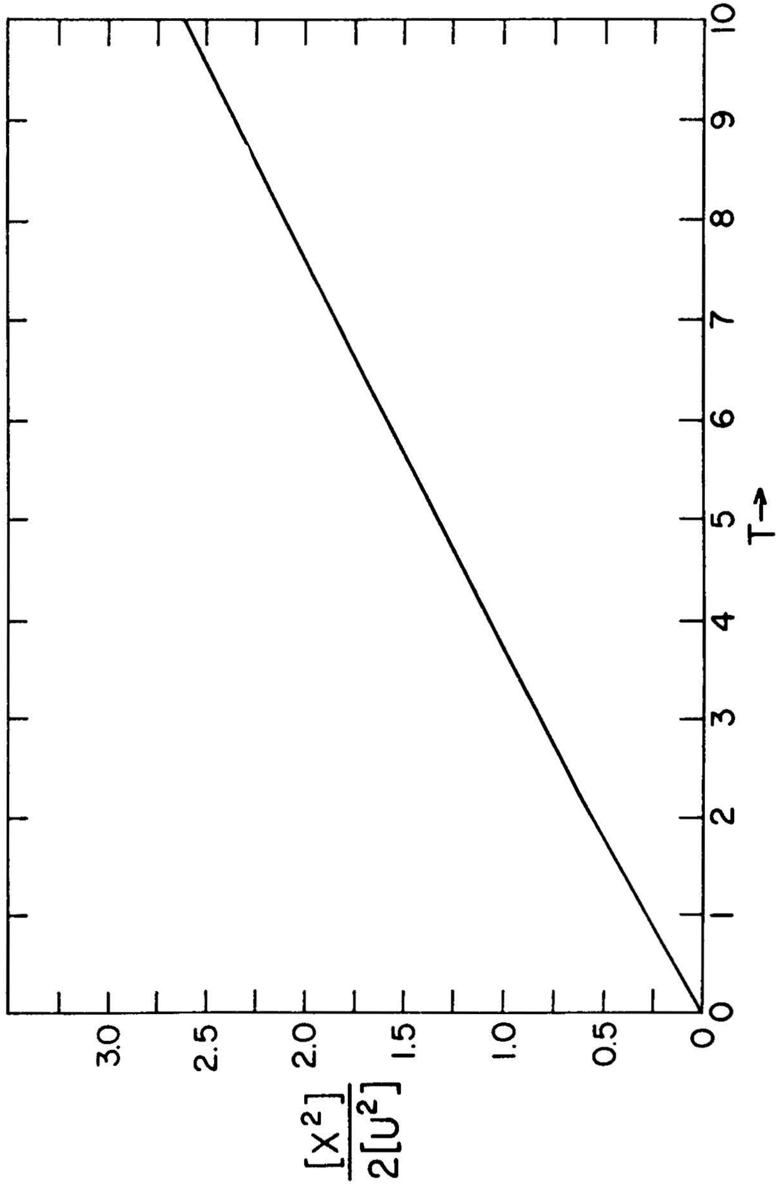

FIGURE 7

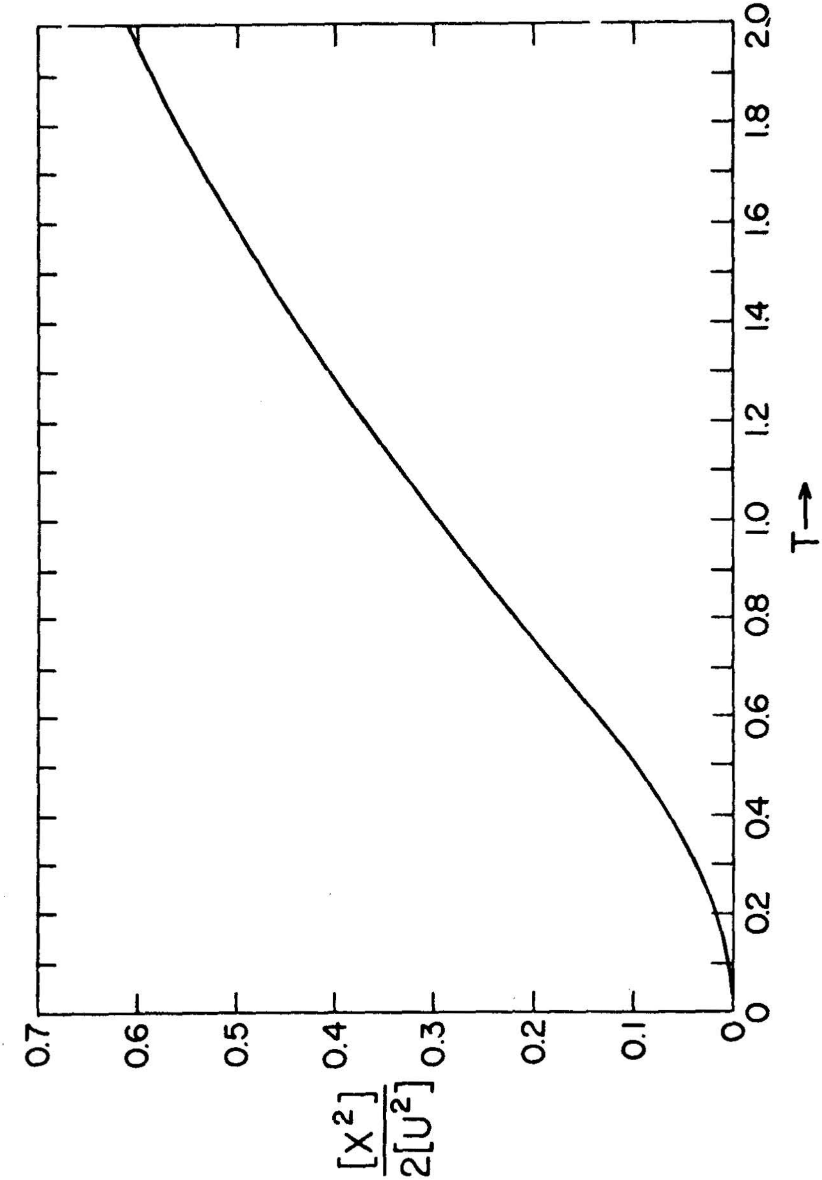

FIGURE 8

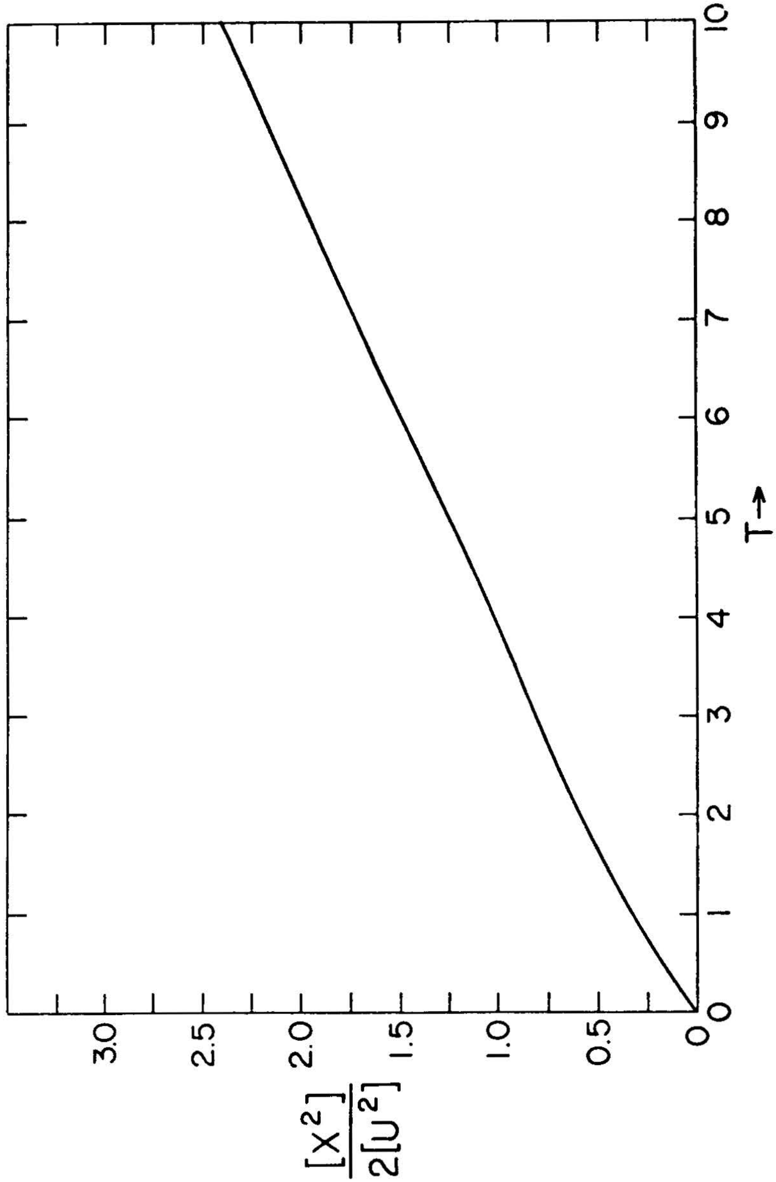

FIGURE 9

RESULTS AND CONCLUSSIONS

Results obtained from this exercise can be briefly summarized.

(1) For the observed experimental values given by Frenzen and shown in figure two of this report, the best fit curve is obtained by the functional representation

$$R(\xi) = e^{-4.48\,\xi^2} \cos 3.905\,\xi^2$$

with a least mean square error F = 0.104.
The functional expression

$$R(\xi) = e^{-1.55\,\xi} \cos 1.88\,\xi$$

yields a least mean square error F = 0.225.

(2) For the observed experimental values given by Frenzen and shown in figure three of this report, the best fit curve is obtained by the functional expression

$$R(\xi) = e^{-1.014\,\xi} \cos 1.836\,\xi$$

with a least mean square error F = 0.176.
The functional expression

$$R(\xi) = e^{-5.70\,\xi^2} \cos 3.603\,\xi^2$$

yields a least mean square error F = 0.603.

(3) The proposed alternative analytical expressions for R (ξ) yield better fits to the experimental data than does that suggested by Frenzen.

(4) A general solution to Taylor's diffusion equation cannot be obtained when analytical expressions for R (ξ) of the type defined in equation (14) are introduced, since the integrals cannot be evaluated analytically.

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TAYLOR, G. I., 1921. Diffusion by continuous movements. Proceedings of the London Mathematical Society, Ser. 2, vol XX, pp. 196-212.