GEOFISICA INTERNACIONAL

COMMUNICATION

THE OPERATION OF SECOND ORDER WAVE MODES ON A UNIFORMLY SLOPING BEACH

E. O. OKEKE*

RESUMEN

Este artículo analiza la teoría referente a los efectos de las ondas de segundo orden, asociadas con las ondas líquidas, a medida que éstas se propagan sobre una playa uniformemente en declive. Se emplea un modelo de playa con contornos profundos, paralelos a la línea de costa. Incidentalmente esto asegura que las soluciones específicas identificadas con las ondas, quedan limitadas.

Con estas consideraciones se demuestra, que los modelos de oscilación de segundo orden son excitados y mantenidos por la energía derivada de las interacciones no lineares entre el perfil de la onda oscilatoria y la velocidad de los componentes de la onda primaria.

Los cálculos numéricos sugieren que las amplitudes de los armónicos de segundo orden parecen crecer constantemente a medida que se propagan hacia la línea de costa, y además aumentan paralelamente al gradiente de playa.

Dentro del ámbito de las frecuencias tidales, la teoría parece predecir una estimación de 1) la anchura del bajío continental sobre el cual los modos, es probable que reproduzcan efectos de fondo; 2) el crecimiento y declinación de los modos tidales como funciones de la distancia de la zona de ruptura.

*Department of Mathematics, University of Benin, Nigeria.

ABSTRACT

This paper addresses the theory concerning second order wave effects associated with water waves as the letter propagate through a uniformly sloping beach. A model beach with bottom contours parallel to the shoreline is used. It is also assumed that there exists a breaker zone in the neighborhood of the shoreline. Incidently, this ensures that the eigensolutions associated with primary wave motions are bounded.

Given the above considerations, it is shown that the second order modes of oscillations are excited and maintained by the energy derived from the non-linear interactions among the oscillatory wave profiles and the velocity of the primary wave components. Furthermore, numerical calculations suggest that the amplitudes of the second order harmonics seem to steadily increase as they propagate towards the shoreline as well as increase with increasing beach gradient. In the range of tidal frequencies, the theory appears to have realistically predicted an estimate of (1) the width of the continental shelf over which propagating modes are likely to experience bottom effects, and (2) the rise and fall of the tidal modes as functions of the distance from the breaker zone.

INTRODUCTION

The phenomena of higher order wave effects associated with water waves have long been identified. Some aspects of these phenomena, characteristic of deep water waves, are well presented in Kinsman (1965). Briefly, it was established that if the analysis of these waves were extended to the third order, there exists a possible tertiary interaction among the component wave modes. However, in connection with shallow water waves, the interaction among the first order waves transfers energy to the second order waves. Thus, in subsequent developments, the latter may obtain significant size which is comparable to the former.

According to Kortweg and De Vries (1895), the development of a realistic non-linear analysis related to shallow water waves should include the effects of the vertical velocity components. It is established that these effects are significant as soon as the analysis is carried beyond the limitations of linearization. As is of the case, we shall, however, attempt to make the problem tractable by utilizing some already established approximations.

These approximations are found in the basic system of non-linear differential equations numbered (1.08) in Stoker (1948). According to Stoker (1948), these equations have been thoroughly tested and were found accurate enough to describe the changes in the shape of water waves propagating in shallow water, up to breaker zone. Thus, they have been widely employed in the study of bores, breakers, tides in canals and in open sea areas (Lamb, 1948).

In the course of this paper, we intend to utilize a simple model of a shallow water zone which is uniformly sloping. A previous study (Darbyshire and Okeke, 1969) suggests that such a beach is associated with a considerable quantity of trapped wave energy and therefore, growth in wave amplitude. Incidentally, the second and higher order wave effects form recognizable factors on such ocean beaches. The developments of wave breaking, formation of bores (Stoker, 1957; Tuck, 1957), higher tidal harmonics (with frequencies which are multiple that of the primary disturbance) are all possible related events.

A REVIEW OF THE SHALLOW WATER WAVE EQUATION

Given time t ($t > 0$) and distance x measured in a direction normal to the shoreline, the water surface profile, η (x, t), and the related x-component of the horizontal velocity, U (x, t), satisfy the following simplified system of non-linear differential equations (Clement, *et al.*, 1975; Lamb, 1945; Stoker, 1948):

$$\frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} + g \frac{\partial}{\partial x} \eta = 0 \tag{2.1}$$

$$\frac{\partial \eta}{\partial t} + \left\{ (\eta + h) \frac{\partial U}{\partial x} + U \frac{\partial}{\partial x} (\eta + h) \right\} = 0 \tag{2.2}$$

where h (x) is the depth of the water layer measured from its undisturbed surface, and g is the acceleration due to gravity. If the non-linear terms are ignored as first approximation, equations (2.1) and (2.2) may be written:

$$\frac{\partial U}{\partial t} + g \frac{\partial}{\partial x} \eta = 0 \tag{2.3}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} (h U) = 0 \tag{2.4}$$

Superimposed on these equations are η_1, and U_1 which are the second order corrections satisfying the following equations:

$$\frac{\partial U_1}{\partial t} + g \frac{\partial \eta_1}{\partial x} = - U \frac{\partial U}{\partial x} \tag{2.5}$$

$$\frac{\partial \eta_1}{\partial t} + \frac{\partial}{\partial x} (h U_1) = - \frac{\partial}{\partial x} (U \eta) \tag{2.6}$$

From equations (2.3) and (2.4) it follows that:

$$g \frac{\partial^2}{\partial x^2} (hU) - \frac{\partial^2}{\partial t^2} U = 0 \tag{2.7}$$

$$\frac{\partial^2}{\partial t^2} \eta - g \frac{\partial}{\partial x}\left(h \frac{\partial}{\partial x} \eta \right) = 0 \tag{2.8}$$

Given equations (2.5) and (2.6) η_1, satisfies the equation:

$$\frac{\partial^2}{\partial t^2} \eta_1 - g \frac{\partial}{\partial x}\left(h \frac{\partial}{\partial x} \eta_1 \right) =$$

$$= \frac{\partial}{\partial x}\left(h U \frac{\partial}{\partial x} U \right) - \frac{\partial^2}{\partial x \partial t} (U \eta) \tag{2.9}$$

The zeros to the right of equation (2.8) are replaced in equation (2.9) by forcing the function built up by interactions among the components of the wave profile and the velocity of the primary waves.

BEACH WITH A UNIFORM SLOPE

The model beach considered in this study is characterized by a straight shoreline defined by $x = 0$. It is reasonable to assume that η can be expressed as follows:

$$\eta(x, t) = \eta_0(x) \cos \sigma t \tag{3.1}$$

where σ is the characteristic frequency of the oscillation. From equation (2.8), η_0 satisfies the equation:

$$x \frac{d^2}{dx^2} \eta_0 + \frac{d\eta_0}{dx} + \frac{\sigma^2}{\alpha g} \eta_0 = 0 \tag{3.2}$$

where α is the constant beach gradient. In general, equation (3.2) may be solved as below:

$$\eta_0(x) = B_0 J_0(\beta\sqrt{x}) + C_0 Y_0(\beta\sqrt{x}) \tag{3.3a}$$

$$\text{where} \quad \beta = 2\sigma / \sqrt{\alpha g} \tag{3.3b}$$

J_0 and Y_0 are Bessel function of zero order, but of first kind and second kind respectively. For the present, we define B_0 and C_0 to be arbritrary constants. Additionally, if we define

$$U(x,t) = U(x) \sin \sigma t \tag{3.3c}$$

then U (x) satisfies the equation:

$$x \frac{d^2 U}{dx^2} + 2 \frac{du}{dx} + \frac{\sigma^2}{\alpha g} U = 0 \tag{3.4}$$

Equation (3.4) is solved by letting $z = \sigma \left(\frac{x}{g}\right)^{1/2}$ and $U = zy$. Substituting,

$$U(x) = x^{-1/2} [A_0 J_1(\beta\sqrt{x}) + C_1 Y_1(\beta\sqrt{x})] \tag{3.5}$$

In equation (3.5), J_1 and Y_1 are Bessel functions of first order, but as before, of first kind and second kind respectively. A_0 and C_1 are assumed to be arbritrary constants.

The presence of the terms Y_0 and Y_1 in equations (3.3a) and (3.5) implies that the amplitudes of the eigensolutions do attain infinite magnitude along the shoreline (defined as x = 0). However, in the subsequent study, we will assume the existence of a breaker zone in the neighborhood of the shoreline. Thus, the heights of the component waves are bounded, hence $C_0 = C_1 = 0$. B_0 may be defined as the constant wave amplitude before breaking takes place.

$$\beta_0 = A_0 \sqrt{h_0/g} \tag{3.6}$$

Equation (3.6) is the approximation generally associated with long waves; h_0 is defined as the constant depth of the water layer above which the rise in its surface is B_0.

SECOND ORDER SOLUTION

From equation (2.9), (3.3) and (3.5):

$$\frac{\partial}{\partial x} \quad h U \frac{\partial}{\partial x} U \quad - \frac{\partial^2}{\partial t \partial x} (U \eta) =$$

$$= - \cos 2 \sigma t \left[\alpha A_0^2 \frac{d}{dx} \left\{ x^{1/2} J_1 (\beta \sqrt{x}) \frac{d}{dx} (x^{-1/2} \,.\right.\right. \tag{4.1}$$

$$\left.\left. . J_1 (\beta \sqrt{x}) \right\} + \sigma^2 B_0^2 \frac{d}{dx} \left\{ x^{-1/2} J_1 (\beta \sqrt{x}) J_0 (\beta \sqrt{x}) \right\} \right]$$

We shall utilize the following properties of the Bessel function:

$$J_1 (\beta x) = - \frac{d}{dx} J_0 (\beta x) \tag{4.2}$$

and,

$$J_1 (\beta x) + \beta \sqrt{x} \frac{d}{dx} J_1 (\beta x) = \beta \sqrt{x}\, J_0 \tag{4.3}$$

In addition, let

$$\xi = \beta \sqrt{x} \tag{4.4}$$

Transforming to the $-\xi$ co-ordinate and using equations (4.2) and (4.4),

$$\frac{\partial}{\partial x}\left(h U \frac{\partial}{\partial x} U\right) - \frac{\partial^2}{\partial t \partial x}(U\eta) =$$
$$= \beta^4 B_0^2 \left\{ \sigma^2 \frac{d}{d\xi}\left\{\frac{1}{\xi}\frac{d}{d\xi} J_0^2(\xi)\right\} - \frac{\alpha g}{2 h_0} \bullet \right.$$
$$\left. \bullet \frac{d}{d\xi}\left[J_1(\xi) \frac{d}{d\xi}\left\{\frac{J_1(\xi)}{\xi}\right\}\right]\right\} \cos 2\sigma t \tag{4.5}$$

$$J_0(\xi) = \left[\frac{2}{\pi\xi}\right]^{1/2} \sin\left[\xi - \frac{\pi}{4}\right] \tag{4.6}$$

and,

$$J_1(\xi) = \left[\frac{2}{\pi\xi}\right]^{1/2} \cos\left[\xi - \frac{\pi}{4}\right] \tag{4.7}$$

From equation (4.6) and (4.7), we may rewrite equation (2.9) as:

$$\frac{d^2}{d\xi^2}\eta_1 + \frac{1}{\xi}\frac{d}{d\xi}\eta_1 + 16\eta_1 = \frac{4 L_0^2}{\pi}\left\{\frac{1}{\xi^2}\left[4\sigma^2 \sin 2\left(\xi - \frac{\pi}{4}\right) - \right.\right.$$
$$\left. - \frac{2\alpha g}{h_0}\cos 2\left(\xi - \frac{\pi}{4}\right)\right] - \frac{1}{\xi^2}\left[8\sigma^2 \sin^2\left(\xi - \frac{\pi}{4}\right) + \right.$$
$$\left.\left. + \frac{5\alpha g}{h_0}\sin 2\left(\xi - \frac{\pi}{4}\right)\right] - \frac{9\alpha g}{h_0 \xi^4}\cos^2\left(\xi - \frac{\pi}{4}\right)\right\} =$$
$$= \frac{4 L_0^2}{\pi}\left\{\cos 2\xi\left[\frac{5\alpha g}{h_0 \xi^2} - \frac{4\sigma^2}{\xi^2}\right] - \sin 2\xi\left[\frac{2\alpha g}{h_0 \xi^2} + \right.\right.$$
$$\left.\left. + \frac{4\sigma^2}{\xi^3} + \frac{\alpha g}{2 h_0 \xi^4}\right] + \left[\frac{9\alpha g}{2 h_0 \xi^4} - \frac{4\sigma^2}{\xi^3}\right]\right\} \tag{4.8}$$

and,

$$L_0^4 = \beta^2 \beta_0^2 \tag{4.9}$$

The power series method of solving the class of equations of which (4.8) is a member, is the asymptotic expansion of the form

$$\eta_1 = C_0 J_0 (4\xi) + L_0^2 Y_1(\xi) \tag{4.10}$$

Where

$$Y_1(\xi) = \sin 2\xi \sum_{r=0}^{\infty} A_r \xi^{-r} + \cos 2\xi \sum_{r=0}^{\infty} B_r \xi^{-r} + \sum_{r=0}^{\infty} C_r \xi^{-r} \tag{4.11}$$

If equation (4.11) is introduced into equation (4.8), the coefficients are determined, yielding:

$$Y_1(\xi) = \left[\frac{2\sigma^2}{\xi} + \frac{5\alpha g}{4h_0 \xi^2} - \frac{\sigma^2}{6\xi^3} + \frac{1}{108\xi^4}\left(4\sigma^2 + \frac{53\alpha g}{2h_0}\right) + \cdots \right] \sin 2\xi - \left[12\sigma^2 - \frac{1}{\xi}\left(\frac{\sigma^2}{3} - \frac{7\alpha g}{2h_0}\right) + \frac{5\sigma^2}{6\xi^2} - \frac{1}{36\xi^3}\left(\sigma^2 - \frac{\alpha g}{h_0}\right) + \cdots \right] \cos 2\xi + \frac{6\sigma^2}{\xi} - \frac{\sigma^2}{8\xi^3} + \frac{9\alpha g}{32h_0\xi^4} - \cdots \tag{4.12}$$

From equations (4.1) and (4.4)

$$\eta_1(x,t) = \eta_1(x) \cos 2\sigma t \tag{4.13}$$

DISCUSSION

In the following calculations, we take $C_0 = 1$. If a swell with a period between 12 and 18 seconds is being considered, we take $L_0 = 0.3$, corresponding to $B_0 = 13$ cm, $\alpha = 0.05$ and $g = 980$ cm sec^{-2}. Likewise, within the tidal range of frequencies, we take $L_0 = 0.036$. for which $B_0 = 3.2$ m.

Furthermore, η_1 consists of two parts, viz: (1) the complementary term represented by the Bessel function with an argument twice that of the primary disturbances, and (2) the particular solution consisting of the positive and negative powers of ξ with the corresponding amplitude which is the square of the primary wave. Numerical calculations performed on the two terms suggest that (1) the solution is finite almost everywhere in the open interval $0 < \xi < \infty$; (2) the amplitude of the rise and fall of the water tends to increase with increasing beach gradient though within physically realistic limits; and (3) in the closed zone $0.05 \leqslant \xi \leqslant 8.0$ which corresponds to $0.7 \leqslant x \leqslant 46.2$ km as measured from the shoreline, the particular integral makes a significant contribution to the solution. However, when $x = 46.2$ km, the sum of the positive and negative terms tends cancel out, hence the contribution of this term is negligible. Similarly, the complementary solution is small outside this zone. This suggests that 46.2 km may be regarded as the width of the model beach considered in this study.

It is noted that the preceding approximations utilized in the model do not give rise to any significant breakdown in the solution. Instead, we have a gradual increase in the heigh of the water levels towards the shore. This phenomenon apparently agrees with the observation made about the advancing water waves over sloping beaches. The change in the form of η_1 as a function of x is ilustrated in figure 1.

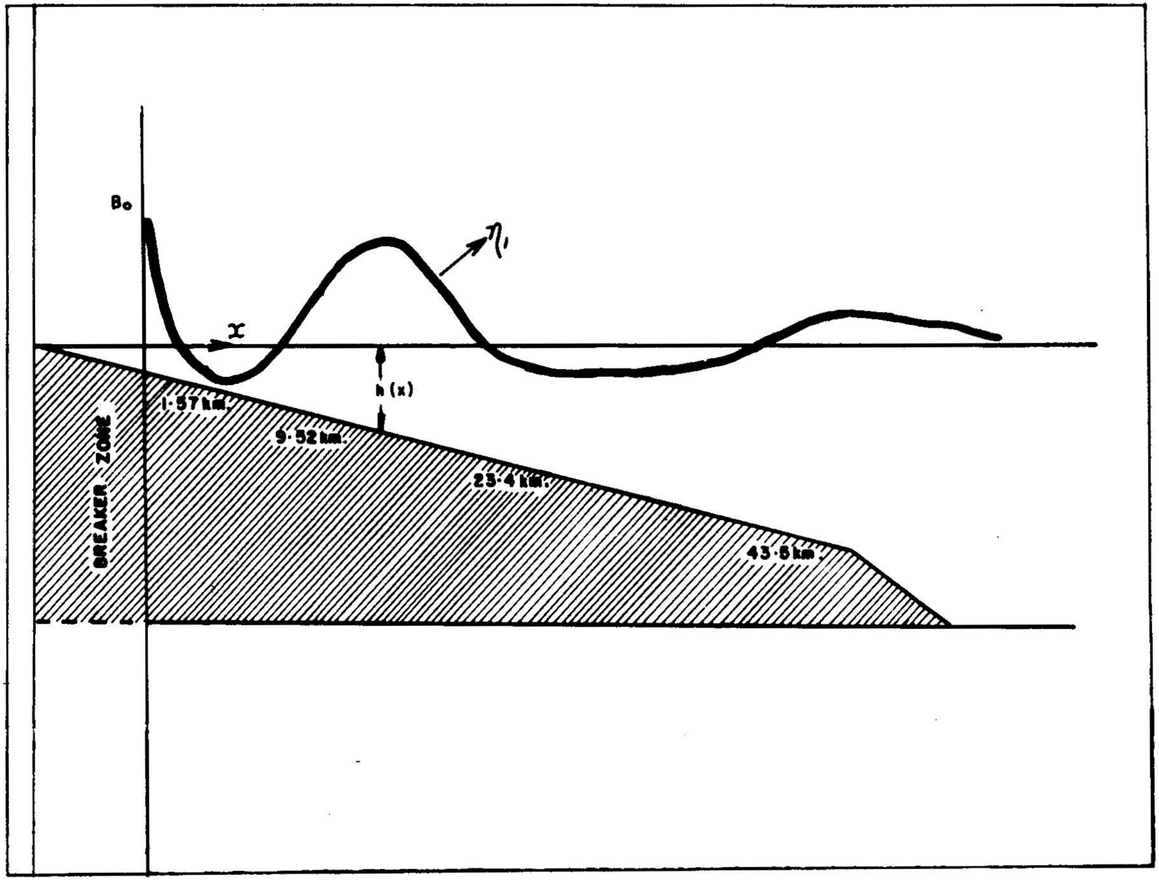

Figura 1. The Profile of η (x) as a function of distance measured from the Breaker Zone.

BIBLIOGRAPHY

CLEMENT, D. L. and C. ROGER, 1975. Analytic solution of the linearized shallo water wave equations for certain continuous depth variations. *J. Austr. Mati Soc.* 19 (Series B), 63-80.

DARBYSHIRE, J. and E. O. OKEKE, 1969. A Study of primary and secondai microseisms recorded in Anglesey. *Geophys. J.R. Astr. Soc.* 17, 63-92.

KINSMAN, B., 1965. *Wind Waves.* Prentice-Hall, Inc., Englewood Cliffs. New Jerse 584-636.

KORWEG, D. G. and G. DE VRIES, 1895. On the change of form of long waves a vancing in rectangular canal and a new type of long stationary waves. *Phil. Ma* (5) 39, 422-443.

LAMB, H., 1948. *Hydrodynamics.* Cambridge University Press. London, 280-283.

STOKER, J. J., 1948. The formation of breakers and bores. Com, *in Pure and Ap Maths.* Vol. 1 N^{o}. 1, 1-35.

STOKER, J. J., 1957. Water Waves. *Inter-Science Publishers Inc.*, New York.

TUCK, E. O. and LI-SAN HWANG, 1972. Long Wave Generation on a sloping bea *J. Fluid Mech.* Vol 51 part. 3, 449-461.