

*FORMULATION OF THE THEORY OF PERTURBATIONS FOR
COMPLICATED MODELS. PART I: THE ESTIMATION OF
THE CLIMATE CHANGE*

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RESUMEN

En este artículo se presenta la construcción de la teoría de las perturbaciones y la formulación de problemas inversos con la finalidad de identificar sus parámetros importantes.

El aspecto novedoso es la extensión de una clase de modelos matemáticos a problemas cuasilineales y no lineales. Las ecuaciones básicas en los modelos se hacen corresponder con las ecuaciones conjugadas que se usan para construir la teoría de perturbaciones. Ejemplos de la teoría de la circulación general de la atmósfera y del clima se presentan.

ABSTRACT

The paper is devoted to construction of the theory of perturbations and formulation of inverse problems for the purpose of identification of its important parameters.

The new aspect here is extension of a class of mathematical models to quasilinear and nonlinear problems. The basic equations in the models are put to correspond with the conjugate equations which are used for constructing the theory of perturbations. Examples from the theory of general circulation of the atmosphere and climate are presented.

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INTRODUCTION

Usually it is very difficult to construct mathematical models simulating complicated processes and phenomena. Such models must incorporate many effects, not all of which are describable with required accuracy. This means that each time we use one or another simplified mathematical formulation of a problem which allows us to describe only a few characteristic features of a process omitting many very important details. However such an approach toward mathematical simulation of physical processes is the basic instrument in our perception of natural phenomena which is being continuously refined.

In the present paper an attempt is made to present a more or less general approach towards construction of such mathematical models and their analysis.

To make our presentation a more explicit one, we will consider as an example the evaluation of the effect of different factors on climate and general circulation of the atmosphere, since this is the central problem in the study of the human environment. The climate of the Earth is known to us from observation. Though the climate of the planet changes, its changes are associated with long-period processes than manifest themselves distinctly after a period of many years. Therefore averaging of the data for many years that characterize climatic functionals of the atmosphere proves to be adequate for both description of the quasistationary climate and construction of the perturbation theory.

At the present time the preliminary evaluation of climatic modifications due to different factors, especially due to industrial human activities is sometimes a more important task than simulation of the proper climate. Therefore in the present paper we try to discuss different approaches to the construction of the perturbation theory with respect to the climate. We will start from the two fundamental statements. First, a system of equations of the atmosphere and ocean dynamics in its most complete form (details unknown) is capable to describe the climate of the planet's atmosphere. The climate is

assumed to be known from observations. Second, perturbations of the climate are regarded to be small. The latter is equivalent to the condition of additivity of the climate and its perturbations. These two assumptions will allow us to formulate mathematical models for simulation of modifications of the climate.

On the whole the perturbation theory and evaluation of functionals of a problem are of a rather general character and as a rule they have very little to do with a specific problem. Therefore results of the present study are applicable to different mathematical models simulating complicated processes in physics, chemistry, biochemistry, engineering, etc.

BASIC AND CONJUGATE EQUATIONS

Let the stationary climate of the atmosphere and the ocean be describable by the system of equations which can be written in the operator form

$$A\varphi = f \quad (1.1)$$

where A is the matrix differential operator dependent on the solution – the vector function φ and the input data $\alpha_1, \alpha_2, \dots, \alpha_n$ – functions of the coordinates, f is the given vector of sources-functions of the coordinates and certain input data $\beta_1, \beta_2, \dots, \beta_m$. Thus, we have $A = A(\varphi, \alpha_1, \alpha_2, \dots, \alpha_n)$ and $f = f(\beta_1, \beta_2, \dots, \beta_m)$.

Let $\bar{\alpha}_1, \bar{\alpha}_2, \dots, \bar{\alpha}_n$ and $\bar{\beta}_1, \bar{\beta}_2, \dots, \bar{\beta}_m$ be input data pertinent to the climate. Employing the above input data, equation (1.1) is written as

$$A(\bar{\varphi}, \bar{\alpha}_1, \dots, \bar{\alpha}_n)\bar{\varphi} = f(\bar{\beta}_1, \dots, \bar{\beta}_m) \quad (1.2)$$

or, assuming

$$\begin{aligned} \bar{A} &= A(\bar{\varphi}, \bar{\alpha}_1, \dots, \bar{\alpha}_n), \\ f &= f(\bar{\beta}_1, \dots, \bar{\beta}_m), \end{aligned} \quad (1.3)$$

equation (1.2) is written as

$$\bar{A} \bar{\varphi} = \bar{f}.$$

We assume that the solution of (1.3) corresponds to the climate obtained empirically on the basis of meteorological and oceanic data.

It will be thought that the input data of problem (1.2) have changed a little and, instead of climatic values $\bar{\alpha}_i, \bar{\beta}_j$ we have

$$\alpha_i = \bar{\alpha}_i + \delta\alpha_i, \beta_j = \bar{\beta}_j + \delta\beta_j,$$

where deviations $\delta\alpha_i$ and $\delta\beta_j$ are assumed to be small against α_i and $\bar{\beta}_j$ respectively.

Then, instead of (1.2) we have

$$A(\bar{\varphi} + \delta\varphi, \bar{\alpha}_i + \delta\alpha_i)(\bar{\varphi} + \delta\varphi) = f(\bar{\beta}_j + \delta\beta_j) \quad (1.4)$$

where the following representation is used:

$$\varphi = \bar{\varphi} + \delta\varphi.$$

In a similar way we *a priori* assume that $\delta\varphi$ is, in a sense, much less than φ . Assuming a sufficient smoothness we consider the expressions

$$\begin{aligned} A(\bar{\varphi} + \delta\varphi, \bar{\alpha}_i + \delta\alpha_i) &= \bar{A} + \frac{\partial \bar{A}}{\partial \bar{\varphi}} \delta\varphi + \frac{\partial \bar{A}}{\partial \bar{\alpha}_i} \delta\alpha_i + \dots \\ f(\bar{\beta}_j + \delta\beta_j) &= \bar{f} + \frac{\partial f}{\partial \bar{\beta}_j} \delta\beta_j + \dots \end{aligned} \quad (1.5)$$

Substituting (1.5) into (1.4) and employing the terms of first order smallness, we obtain

$$\bar{A} \bar{\varphi} + (\bar{A} + \frac{\partial \bar{A}}{\partial \bar{\varphi}} \bar{\varphi}) \delta\varphi + \frac{\partial \bar{A}}{\partial \bar{\alpha}_i} \bar{\varphi} \delta\alpha_i = \bar{f} + \frac{\partial \bar{f}}{\partial \bar{\beta}_j} \delta\beta_j. \quad (1.6)$$

With equation (1.3) we have

$$(\bar{A} + \frac{\partial \bar{A}}{\partial \bar{\varphi}} \bar{\varphi}) \delta \varphi = \frac{\partial \bar{f}}{\partial \bar{\beta}_j} \delta \beta_j - \frac{\partial \bar{A}}{\partial \alpha_i} \bar{\varphi} \delta \alpha_i \quad (1.7)$$

which is the basic equation for definition of small deviations of the solution φ from climate.

We assume now that the operator of the climatic problem \bar{A} and the source of the climate \bar{f} are known with limited accuracy, i.e. let

$$\begin{aligned} \bar{A} &= \bar{A} + \epsilon, \\ \bar{f} &= \bar{F} + \xi, \end{aligned} \quad (1.8)$$

where ϵ is the operator of the model's error

$$\epsilon = \bar{A} - \bar{A},$$

and ξ is the error in the vector function of the sources

$$\xi = \bar{f} - \bar{F}.$$

Let

$$\begin{aligned} \| A \| &\gg \| \epsilon \|, \\ \| f \| &\gg \| \xi \|. \end{aligned} \quad (1.9)$$

Here the norm of the operator and that of the vector function are consistent with metric space.

Substituting (1.8) into (1.7) we have

$$(\bar{A} + \frac{\partial \bar{A}}{\partial \bar{\varphi}} \bar{\varphi}) \delta \varphi = \frac{\partial \bar{F}}{\partial \bar{\beta}_j} \delta \beta_j - \frac{\partial \bar{A}}{\partial \alpha_i} \bar{\varphi} \delta \alpha_i + \eta, \quad (1.10)$$

where

$$\eta = \frac{\partial \xi}{\partial \bar{\beta}_j} \delta \beta_j - \frac{\partial \epsilon}{\partial \bar{\alpha}_i} \bar{\varphi} \delta \alpha_i - \left(\epsilon + \frac{\partial \epsilon}{\partial \bar{\varphi}} \bar{\varphi} \right) \delta \varphi. \quad (1.11)$$

Thus, under assumption (1.9), equation (1.10) is written as follows:

$$\begin{aligned} \left(\bar{\Lambda} + \frac{\partial \bar{\Lambda}}{\partial \bar{\varphi}} \bar{\varphi} \right) \delta \varphi = & \frac{\partial \bar{F}}{\partial \bar{\beta}_j} \delta \beta_j - \frac{\partial \bar{\Lambda}}{\partial \bar{\alpha}_i} \bar{\varphi} \delta \alpha_i \quad (1 + \\ & + O(\| \epsilon \| + \| \xi \|)) \end{aligned}$$

Hence, accurate to small quantities:

$$\left(\bar{\Lambda} + \frac{\partial \bar{\Lambda}}{\partial \bar{\varphi}} \bar{\varphi} \right) \delta \varphi = \frac{\partial \bar{F}}{\partial \bar{\beta}_j} \delta \beta_j - \frac{\partial \bar{\Lambda}}{\partial \bar{\alpha}_i} \bar{\varphi} \delta \alpha_i. \quad (1.12)$$

The above equation is a model one for calculation of modifications of climate subject to variations of input data $\delta \alpha_i$ and $\delta \beta_j$. We introduce notation

$$\begin{aligned} L + \bar{\Lambda} + \frac{\partial \bar{\Lambda}}{\partial \bar{\varphi}} \bar{\varphi}, \\ \delta F = \frac{\partial \bar{F}}{\partial \bar{\beta}_j} \delta \beta_j - \frac{\partial \bar{\Lambda}}{\partial \bar{\alpha}_i} \bar{\varphi} \delta \alpha_i, \end{aligned} \quad (1.13)$$

then, finally, we have

$$L \delta \varphi = \delta F \quad (1.14)$$

and the formal solution is

$$\delta \varphi = L^{-1} \delta F. \quad (1.15)$$

CONJUGATE EQUATIONS AND THE THEORY OF PERTURBATIONS

Formula (1.15) of the theory of perturbations is good for the case when it is necessary to evaluate deviations from the climatic solution for only one set of variations of input data. In the planning of purposeful modification of climate under the influence of human activity it is important to make a series of test calculations. If we take into account the fact that in future we shall have to solve problems on optimal climate control, the deficiencies of the above approach become evident. Therefore in this part we shall try to build a more or less universal theory of perturbations for given functionals of problems which will allow us to decrease essentially the amount of computation. The theory makes use of conjugate equations.

Along with the basic operator equation

$$L\delta\varphi = \delta F \quad (2.1)$$

we introduce into consideration the conjugate equation

$$L^*\varphi^* = p, \quad (2.2)$$

where L and L^* are conjugate operators in the sense of Lagrange.

$$(Lg, h) = (g, L^*h). \quad (2.3)$$

Here g and h are elements of Hilbert space from the domain of definition of operators L and L^* respectively. No assumptions are made for the function p , as yet.

Let us multiply scalarly (2.1), (2.2) by φ^* and $\delta\varphi$ respectively and subtract one result from the other. Then

$$(\varphi^*, L\delta\varphi) - (\delta\varphi, L^*\varphi^*) = (\varphi^*, \delta F) - (p, \delta\varphi). \quad (2.4)$$

The left-hand side of (2.4), according to (2.3), turns into zero. Hence

$$(p, \delta\varphi) = (\varphi^*, \delta F) \quad (2.5)$$

Let us consider a set of linear functionals

$$\varphi J_n = (p_n, \delta\varphi). \quad (2.6)$$

If, in particular, $p_n = \delta(\underline{x} - \underline{x}_n)$, then

$$\delta J_n = \delta\varphi(\underline{x}_n).$$

The function φ^* , corresponding to p_n , will be denoted by φ_n^* .

Thus, on the basis of (2.5), we have a series of functionals

$$\delta J_n = (\varphi_n^*, \delta F). \quad (2.7)$$

Suppose that we have chosen N functionals J_1, J_2, \dots, J_N . C in advance. For this purpose we solve N conjugate problems

$$L^* \varphi_n^* = p_n \quad (n = 1, 2, \dots, N). \quad (2.8)$$

Let problems (2.8) be solved. Then, with the help of (2.4) it is possible to find N functionals or, using the notation (1.13),

$$\delta J_n = (\varphi_n^*, \frac{\partial \bar{F}}{\partial \bar{\beta}_j} \delta \beta_j) - (\varphi_n^*, - \frac{\partial \bar{L}}{\partial \bar{\alpha}_i} \bar{\varphi} \delta \alpha_i),$$

$$(n = 1, 2, \dots, N). \quad (2.9)$$

From (2.9) it follows that we need not calculate variations $\delta\varphi$, corresponding to different sets of parameters $\delta\alpha_i$ and $\delta\beta_j$, since, given φ_n^* ($n = 1, \dots, N$), we can directly compute functionals δJ_n for any sets of variations of input data.

The next task is to solve the system of equations obtained.

THE THEORY OF PERTURBATIONS FOR NONSTATIONARY PROBLEMS

Let us consider now a nonstationary problem in an abstract form

$$\frac{\partial \varphi}{\partial t} + A\varphi = f, \quad \varphi = g \quad \text{at } t = 0. \quad (3.1)$$

which is put to correspond with

$$-\frac{\partial \varphi^*}{\partial t} + A^*\varphi^* = f^*, \quad \varphi^* = g^* \quad \text{at } t = T. \quad (3.2)$$

Here f^* and g^* are as yet undefined vector functions to be chosen later. Equations (3.1), (3.2) are, respectively, multiplied by φ^* , φ , the results are subtracted one from the other and integrated over t on the interval $0 \leq t \leq T$. As a result we have

$$\begin{aligned} \int_0^T \frac{\partial}{\partial t} (\varphi^*, \varphi) dt + \int_0^T dt [(\varphi^*, A\varphi) - (\varphi, A^*\varphi^*)] = \\ \int_0^T dt [(f, \varphi^*) - (f^*, \varphi)]. \end{aligned} \quad (3.3)$$

Since A^* and A are conjugate operators

$$(\varphi^*, A\varphi) - (\varphi, A^*\varphi^*) = 0$$

expression (3.3) with consideration of initial conditions, reduces to

$$(g^*, \varphi_T) - (g, \varphi_0) = \int_0^T dt [(f, \varphi) - (f^*, \varphi)], \quad (3.4)$$

where

$$\varphi_T^* = \varphi^*|_{t=T}, \quad \varphi_0 = \varphi|_{t=0}.$$

We now suppose that it is required to calculate the linear functional of solution (3.1) which can be presented as

$$J = (g^*, \varphi_T) + \int_0^T (f^*, \varphi) dt. \quad (3.5)$$

With the help of identity (3.3) this functional is written as

$$J = (g, \varphi_0^*) + \int_0^T (f, \varphi^*) dt. \quad (3.6)$$

Assuming that the input data of (3.1) are somewhat perturbed, i.e. in place of g and f , we consider $g' = g + \delta g$ and $f' = f + \delta f$. Then, on the basis of (3.6), we obtain the variation of the functional

$$\delta J = (\delta g, \varphi_0^*) + \int_0^T (\delta f, \varphi^*) dt. \quad (3.7)$$

Hence, for evaluation of variations of functional J depending on different variations of data, it is not necessary to solve many problems of type (3.1):

$$\frac{\partial \varphi'}{\partial t} + A\varphi' = f', \quad \varphi' = g' \quad \text{at} \quad t = 0 \quad (3.8)$$

with various f' and g' . It is enough to solve only one conjugate problem (3.2) and employ formula (3.7).

By means of formula (3.7) one can state the inverse problem of finding δg and δf for the set of functionals δJ .

The above methods of the perturbation theory were based, to an extent, on the employment of conjugate equations of the theory of climate.

This is natural since the conjugate equations in this interpretation define the domain of influence of variations of input parameters, over the whole space, on the variations of the functional of the solution in a given region. Therefore the study of conjugate equations and understanding of general principles of climate modification on this basis is an important task.

However, this is not the only way to evaluate climate variations. There is a simpler approach of direct integration of equations of climate. If the modification of climate proves to be essential because of the variation of input parameters then it is required to solve initial problems corresponding to different sets of input data.

However, when evaluating the effect of industrial human activities on climate from the global point of view, we often have to deal with insignificant modification of climate. In this case it is more natural to employ the equations that are linearized with respect to climate. Then we arrive at problem (1.14) in a stationary case

$$L\delta\varphi = \delta F \quad (3.9)$$

in a nonstationary case

$$\frac{\partial \delta\varphi}{\partial t} + L\delta\varphi = \delta F. \quad (3.10)$$

As a result we arrive at the solutions which make it possible to estimate the modification of climate in different points of the global system. Naturally such solutions can be used for obtaining the above mentioned functionals. It is appropriate to stress here which methods of calculation of climate variations should be used in one or another case. Thus if we calculate climate modifications in many regions of the Earth under the influence of the same variations of input data it is most efficient to use the solution in the form (3.9), (3.10). If we concentrate attention only on some functionals of the problems and pre-calculate variations of those functionals with respect to different variations of input data it is best to employ the theory of perturbations and conjugate equations, since for this purpose it is necessary to solve conjugate problems, corresponding to a set of functionals, only once, and in an explicit form to evaluate the solutions through variations of climate parameters.

Thus the two above mentioned approaches supplement each

other and make it possible to get a deeper insight into certain climate modifications.

The approach which uses conjugate equations for estimation of climate modification is especially valuable when we deal with the planning of economic development of regions with regard to the results of industrial human activity. In this case, for instance, there is a possibility to formulate explicitly a problem of optimal location of industrial enterprises with respect to the given functionals that take account of the preservation of the environment in residential zones. These questions will be considered below.

SPECTRAL METHOD AND THE THEORY OF PERTURBATIONS

When considering the fundamental features of modification of the Earth's climate it is sometimes convenient to make use of the spectral formulation of problems of the dynamics of the atmosphere and ocean and on its basis to formulate the theory of perturbations.

For this purpose let us start from equation (1.14)

$$L\delta\varphi = \delta F \quad (4.1)$$

and the spectral problem

$$Lw = \lambda w. \quad (4.2)$$

Since L is not a self-conjugate operator we consider, along with (4.2), the following conjugate problem

$$L^*w^* = \lambda w^*, \quad (4.3)$$

where L^* is the conjugate L operator in the sense of Lagrange.

$$(w^*, Lw) = (w, L^*w^*). \quad (4.4)$$

Let problem (4.2) define a complete system of eigenfunctions $\{w_n\}$ and a corresponding system of eigenvalues $\{\lambda_n\}$, and problem (4.3) that of $\{w_n^*\}$ and $\{\lambda_n\}$. Then on the basis of (4.4) orthogonalities take place, so that

$$(w_n, w_m^*) = 0, \text{ if } n \neq m.$$

As a result it is not difficult to arrive at the condition of biorthogonality

$$(w_n, w_m^*) = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{if } n \neq m. \end{cases} \quad (4.5)$$

The solution of (4.1) is sought as a Fourier series according to the eigenfunctions of the spectral problem (4.2):

$$\delta\varphi = \sum_n \delta\varphi_n w_n \quad (4.6)$$

and, similarly,

$$\delta F = \sum_n \delta F_n w_n, \quad (4.7)$$

where

$$\delta\varphi_n = (\delta\varphi, w_n^*), \quad \delta F_n = (\delta F, w_n^*).$$

Substituting (4.6) and (4.7) into (4.1) and scalarly multiplying the result by w_m^* we obtain the equalities

$$\lambda_m \delta\varphi_m = \delta F_m \quad (m = 1, 2, \dots). \quad (4.8)$$

Thus the solution of problem (4.1) is written as

$$\delta\varphi = \sum_n \frac{\delta F_n}{\lambda_n} w_n \quad (4.9)$$

or, in final form,

$$\delta\varphi = \sum_n \frac{(\delta F, w_n^*)}{\lambda_n} w_n. \quad (4.10)$$

In this way the perturbation of the solution from the basic state is obtained in the form of a Fourier series (4.9) or (4.10) by means of perturbation of the input data of the problem δF .

If one is interested in linear functional rather than the perturbation $\delta\varphi$, i.e.

$$\delta J = (p, \delta\varphi), \quad (4.11)$$

in this case

$$\begin{aligned} \delta J &= \sum_n \frac{(\delta F, w_n^*)}{\lambda_n} (p, w_n) = \\ &= (\delta F, \varphi^*), \end{aligned} \quad (4.12)$$

where

$$\delta^* = \sum_n \frac{(p, w_n)}{\lambda_n} w_n^*. \quad (4.13)$$

Here φ^* is a solution to problem (2.2)

$$L^*\varphi^* = p.$$

CONJUGATE EQUATIONS OF THE DYNAMICS OF THE ATMOSPHERE

Let us consider the system of equations of the dynamics of atmospheric processes in adiabatic approximation and study the structure

of the operator of the problem. We will investigate a simple case of the barotropic atmosphere. Then we have a problem

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - lv + RT \frac{\partial \varphi}{\partial x} &= 0, \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + lu + RT \frac{\partial \varphi}{\partial y} &= 0, \\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0. \end{aligned} \tag{5.1}$$

Here u, v are the components of the velocity vector along the coordinate axes (x,y) , where y and x are directed to the North and to the East respectively and $RT = \text{const}$, $\varphi(x, y, z)$ denotes the relative deviation of pressure from the standard one, l is the Coriolis parameter. The square D is supposed to be the domain of definition of the solution. The conditions of periodicity are assumed to be set on the boundary ∂D . Let us consider the solution vector and the matrix

$$\varphi = \begin{pmatrix} u \\ v \\ RT\varphi \end{pmatrix}, \quad A = \begin{pmatrix} \Lambda & -1 & \frac{\partial}{\partial x} \\ 1 & \Lambda & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \end{pmatrix}.$$

Using the following notation of the operator

$$\Lambda = \frac{\partial}{\partial x} u \cdot + \frac{\partial}{\partial y} v$$

we have

$$\Lambda u = \text{div } uu, \quad \Lambda v = \text{div } uv.$$

Then the above system of equations (5.1) can be written in the operator form

$$B \frac{\partial \varphi}{\partial t} + A\varphi = 0, \quad (5.2)$$

where B is the matrix

$$B = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{vmatrix}.$$

We introduce the scalar product in Hilbert space Φ by

$$(g, h)_D = \sum_{i=1}^3 \int_D g_i h_i dD.$$

Here g_i and h_i are the components of the vector-functions g and h , respectively.

Now let us find the conjugate operator with respect to A . For this purpose we consider the Lagrangian identity

$$(g, A Ah)_D = (A^* g, h)_D$$

or

$$(g, Ah)_D = \int_D [(\Lambda u + lv + R\bar{T} \frac{\partial \varphi}{\partial x})u^* + (lu + \Lambda v + R\bar{T} \frac{\partial \varphi}{\partial y})v^* + (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) R\bar{T}\varphi^*] dD. \quad (5.3)$$

For simplicity we take

$$h = \begin{vmatrix} u \\ v \\ R\bar{T}\varphi \end{vmatrix}, \quad g = \begin{vmatrix} u^* \\ v^* \\ R\bar{T}\varphi^* \end{vmatrix}.$$

After integrating by parts, assuming periodicity of the solutions, and after some explicit transformations the expression on the right in (5.3) reduces to

$$(g, Ah)_D = \int_D [(\Lambda^*u^* + lv^* - R\bar{T} \frac{\partial \varphi^*}{\partial x})u + (-lu^* + \Lambda^*v^* - R\bar{T} \frac{\partial \varphi^*}{\partial y})v - (\frac{\partial u^*}{\partial x} + \frac{\partial v^*}{\partial y})R\bar{T}\varphi]dD = (A^*g, h)_D, \quad (5.4)$$

where

$$\Lambda^* = - (\frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v) = - (u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}) = - \Lambda. \quad (5.5)$$

Here we made use of the fact that u and v satisfy the equation of continuity $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$. Taking account of (5.4) and (5.5) we come to

$$\varphi^* = \begin{vmatrix} u^* \\ v^* \\ R\bar{T}\varphi^* \end{vmatrix}, \quad A^* = \begin{vmatrix} -\Lambda & 1 & -\frac{\partial}{\partial x} \\ -1 & -\Lambda & -\frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & -\frac{\partial}{\partial y} & 0 \end{vmatrix} = - A.$$

Until now it has been assumed that u and v are given functions of x , y and time. This assumption can be left out now. Indeed, let us assume that we deal with a quasi-linear system

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - lv + R\bar{T} \frac{\partial \varphi}{\partial x} = 0, \quad (5.6)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + lv + R\bar{T} \frac{\partial \varphi}{\partial y} = 0, \quad (5.6)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and have found the solution of this system under condition of periodicity (5.7) on the boundary and the initial data

$$u = u_0, v = v_0 \quad \text{at} \quad t = 0. \quad (5.7)$$

Regarding functions u and v as coefficients of operators A and A^* we have

$$A = \left\| \begin{array}{ccc} \Lambda & -1 & \frac{\partial}{\partial x} \\ 1 & \Lambda & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \end{array} \right\|, \quad A^* = -A,$$

where operator Λ has now the form

$$\Lambda = \frac{\partial}{\partial x} u \cdot + \frac{\partial}{\partial y} v \cdot.$$

Along with problem (5.6) we introduce the conjugate problem

$$-\frac{\partial u^*}{\partial t} - u \frac{\partial u^*}{\partial x} - v \frac{\partial u^*}{\partial y} + lv^* = R\bar{T} \frac{\partial \varphi^*}{\partial x} = 0, \quad (5.8)$$

$$-\frac{\partial v^*}{\partial t} - u \frac{\partial v^*}{\partial x} - v \frac{\partial v^*}{\partial y} - lu^* = R\bar{T} \frac{\partial \varphi^*}{\partial y} = 0,$$

$$-\frac{\partial u^*}{\partial x} - \frac{\partial v^*}{\partial y} = 0$$

under the condition

$$u^* = u_T^*, v^* = v_T^* \quad \text{at} \quad t = T. \quad (5.9)$$

Problems (5.6), (5.7) and (5.8), (5.9) are presented in the operator form

$$B \frac{\partial \varphi}{\partial t} + A\varphi = 0, \quad (5.10)$$

$$B\varphi = B\varphi_0 \quad \text{at} \quad t = 0$$

and

$$-B \frac{\partial \varphi^*}{\partial t} - A\varphi^* = 0, \quad (5.11)$$

$$B\varphi^* = B\varphi_T^* \quad \text{at} \quad t = T.$$

After multiplying equations (5.10), (5.11) scalarly by φ^* and φ respectively, and then subtracting the result, we come to

$$\frac{d}{dt} (B\varphi, \varphi^*) = 0. \quad (5.12)$$

By integrating these equations under the given conditions at $t = 0$ and $t = T$, we have

$$(B\varphi_T, \varphi_T^*)_D = (B\varphi_0, \varphi_0^*)_D. \quad (5.13)$$

This condition will be used later, and now we rewrite it in the component form

$$\int_D (u_T u_T^* + v_T v_T^*) dD = \int_D (u_o u_o^* + v_o v_o^*) dD. \quad (5.14)$$

It should be noted that if we choose u_T^* and v_T^* as u_T and v_T , we come to the law of conservation of kinetic energy:

$$\int_D E_T dD = \int_D E_o dD.$$

In this case we have the full solution reversibility. This means that after solving problem (5.6), (5.7) and putting $u_T^* = u_T$, $v_T^* = v_T$, we can solve problem (5.8), (5.9) in the reverse direction with respect to time. As a result we come to the same solutions of the basic system as we do in the case of the basic problem.

In conclusion we will show, that sometimes it is preferable to make use of more common phase space $D \times D_t$ with scalar product

$$(g, h)_{D \times D_t} = \sum_{i=1}^3 \int_0^T dt g_i h_i.$$

Introducing operators

$$M = B \frac{\partial}{\partial t} + A$$

and

$$M^* = -B \frac{\partial}{\partial t} - A$$

it is not difficult to see that

$$(M\varphi, \varphi^*)_{D \times D_t} = (\varphi, M^* \varphi^*)_{D \times D_t} - (B\varphi_T, \varphi_T^*)_D + (B\varphi_o, \varphi_o^*)_D \quad (5.15)$$

holds. Taking account of (5.13) we finally have

$$(M\varphi, \varphi^*)_{D \times D_t} = (\varphi, M^* \varphi^*)_{D \times D_t}, \quad (5.16)$$

where

$$M^* = -M.$$

Now let us consider the system of the initial equations with viscosity, i.e. equations

$$\begin{aligned} \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \nu \Delta u + R\bar{T} \frac{\partial \varphi}{\partial x} - \mu \Delta u &= 0, \\ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \nu \Delta v + R\bar{T} \frac{\partial \varphi}{\partial y} - \mu \Delta v &= 0, \end{aligned} \quad (5.17)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

under the condition

$$u = u_0, \quad v = v_0 \quad \text{at} \quad t = 0 \quad (5.18)$$

and the assumption of periodic character of solutions. Then, making use of the above method and applying

$$\Lambda = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} - \mu \Delta$$

and

$$\Lambda^* = -u \frac{\partial}{\partial x} - v \frac{\partial}{\partial y} - \mu \Delta,$$

we come to the system of conjugate equations:

$$\begin{aligned}
 -\frac{\partial u^*}{\partial t} - u \frac{\partial u^*}{\partial x} - v \frac{\partial u^*}{\partial y} + l v^* - R\bar{T} \frac{\partial \varphi^*}{\partial x} - \mu \Delta u^* &= 0, \\
 -\frac{\partial v^*}{\partial t} - u \frac{\partial v^*}{\partial x} - v \frac{\partial v^*}{\partial y} - l u^* - R\bar{T} \frac{\partial \varphi^*}{\partial y} - \mu \Delta v^* &= 0, \\
 -\frac{\partial u^*}{\partial x} - \frac{\partial v^*}{\partial y} &= 0
 \end{aligned} \tag{5.19}$$

provided that

$$u^* = u_T^*, v^* = v_T^* \quad \text{at} \quad t = T. \tag{5.20}$$

The analysis of problems (5.17), (5.18) and (5.19), (5.20) shows that the basic problem should be solved with t increasing in the interval $0 \leq t \leq T$, while the conjugate problem is solved with t decreasing in the interval $T \geq t \geq 0$. Only such computation is correct for each problem which is due to the presence of viscosity forces in equations. From our analysis of perturbation formulas given below it will become clear why conjugate problems have been introduced.

CONJUGATE EQUATIONS FOR BAROCLINIC ATMOSPHERE

Now let us consider the model baroclinic atmosphere in adiabatic approximation

$$\begin{aligned}
 \frac{\partial \bar{\rho} u}{\partial t} + \Lambda u - l \bar{\rho} v + \bar{p} \frac{\partial \varphi}{\partial x} &= 0, \\
 \frac{\partial \bar{\rho} v}{\partial t} + \Lambda v + l \bar{\rho} u + \bar{p} \frac{\partial \varphi}{\partial y} &= 0,
 \end{aligned} \tag{6.1}$$

$$-g\bar{\rho}\vartheta + \bar{p} \frac{\partial \varphi}{\partial z} = 0, \quad (6.1)$$

$$\frac{\partial \bar{\rho}u}{\partial x} + \frac{\partial \bar{\rho}v}{\partial y} + \frac{\partial \bar{\rho}w}{\partial z} = 0,$$

$$\frac{\partial \bar{\rho}\vartheta}{\partial t} + \Lambda\vartheta + \frac{\gamma_a - \gamma}{\bar{T}} \bar{\rho}w = 0$$

under the condition

$$\begin{aligned} \bar{\rho}w &= 0 & \text{at} & \quad z = 0, \\ \bar{\rho}w &= 0 & \text{at} & \quad z = H. \end{aligned} \quad (6.2)$$

Here u, v, w , are the components of the velocity vector along the coordinate axes (x, y, z) , the axis y being directed to the North the axis x – to the East, z – vertically upwards $\varphi = \frac{p'}{\bar{p}}, v = \frac{T'}{\bar{T}}$, where p', T' are deviations of pressure and temperature from the standard values $\bar{p}(z), \bar{T}(z)$, where $\bar{p} = R\bar{\rho}\bar{T}$. The solution is assumed to be periodic in the plane (x, y) and satisfy the initial data

$$u = u_0, v = v_0, \vartheta = \vartheta_0 \quad \text{at} \quad t = 0. \quad (6.3)$$

Let us then assume, that $R\bar{T} = \text{const}, \frac{\gamma_a - \gamma}{\bar{T}} = \text{const}$. The operator is defined by the formula

$$\Lambda = \frac{\partial}{\partial x} \bar{\rho}u + \frac{\partial}{\partial y} \bar{\rho}v + \frac{\partial}{\partial z} \bar{\rho}w.$$

Consequently,

$$\begin{aligned} \Lambda u &= \text{div } \bar{\rho}uu, \quad \Lambda v = \text{div } \bar{\rho}uv, \\ \Lambda \vartheta &= \text{div } \bar{\rho}u\vartheta \end{aligned}$$

Introducing vector -- function φ which is the solution of the problem and the matrix

$$\varphi = \begin{pmatrix} u \\ v \\ w \\ \varphi \\ \vartheta \end{pmatrix}, \quad A = \begin{pmatrix} \Lambda & -\bar{\rho}l & 0 & \bar{p} \frac{\partial}{\partial x} & 0 \\ \bar{\rho}l & \Lambda & 0 & \bar{p} \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & \bar{p} \frac{\partial}{\partial z} & -g\bar{\rho} \\ \frac{\partial}{\partial x} \bar{\rho} & \frac{\partial}{\partial y} \bar{\rho} & \frac{\partial}{\partial z} \bar{\rho} & 0 & 0 \\ 0 & 0 & g\bar{\rho} & 0 & \frac{\bar{T}_g}{\gamma_a - \gamma} \Lambda \end{pmatrix},$$

$$B\varphi_0 = \begin{pmatrix} \bar{\rho}u_0 \\ \bar{\rho}v_0 \\ 0 \\ 0 \\ \frac{\bar{T}_g}{\gamma_a - \gamma} \bar{\rho}\vartheta_0 \end{pmatrix}, \quad B = \begin{pmatrix} \bar{\rho} & 0 & 0 & 0 & 0 \\ 0 & \bar{\rho} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\bar{T}_g}{\gamma_a - \gamma} \bar{\rho} \end{pmatrix}.$$

we can write problems (6.1), (6.3) in the form

$$B \frac{\partial \varphi}{\partial t} + A\varphi = 0,$$

$$B\varphi = B\varphi_0 \quad \text{at} \quad t = 0. \quad (6.4)$$

It is assumed here that the solution belongs to Hilbert space of absolutely continuous and differential functions, satisfying the boundary (6.2) and periodicity conditions. The scalar product is introduced by

$$(g, h)_D = \sum_{i=1}^5 \int_D g_i h_i dD.$$

Let us consider the operator A and find the conjugate one making use of Lagrange identity. As a result of calculations, analogous to the above ones in paragraph 5 we have

$$\varphi^* = \begin{pmatrix} u^* \\ v^* \\ w^* \\ \varphi^* \\ \vartheta^* \end{pmatrix}, \quad A^* = \begin{pmatrix} -\Lambda & \bar{\rho}l & 0 & -\bar{p} \frac{\partial}{\partial x} & 0 \\ -\bar{\rho}l & -\Lambda & 0 & -\bar{p} \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & -\bar{p} \frac{\partial}{\partial z} & g\bar{\rho} \\ -\frac{\partial}{\partial x} \bar{\rho} & -\frac{\partial}{\partial y} \bar{\rho} & -\frac{\partial}{\partial z} \bar{\rho} & 0 & 0 \\ 0 & 0 & -g\bar{\rho} & 0 & -\frac{\bar{T}_g}{\gamma_a - \gamma} \Lambda \end{pmatrix}$$

While constructing the conjugate operator we made use of the fact (which can be easily tested) that the expression

$$\begin{aligned} & \int_D (u^* \operatorname{div} \bar{\rho}uu + v^* \operatorname{div} \bar{\rho}uv + w^* \operatorname{div} \bar{\rho}uw) dD = \\ & = - \int_D (u \operatorname{div} \bar{\rho}uu^* + v \operatorname{div} \bar{\rho}uv^* + w \operatorname{div} \bar{\rho}uw^*) dD \end{aligned}$$

takes place. This relation holds true after fulfilling the following conditions: the assumption that

$$\operatorname{div} \bar{\rho}u = 0,$$

$$\operatorname{div} \bar{\rho}u^* = 0$$

and the requirements that components of the solution φ^* satisfy the conditions of smoothness, the limited relations

$$\begin{aligned}\bar{\rho}w^* &= 0 & \text{at} & \quad z = 0, \\ \bar{\rho}w^* &= 0 & \text{at} & \quad z = H\end{aligned}\tag{6.5}$$

and, finally, the conditions of periodicity of solutions in the plane (x , y). We see that in this case we have

$$A^* = -A.\tag{6.6}$$

Thus, the operator A is skewsymmetric. Our task is to construct conjugate equations corresponding to evolution problems. To this aim we introduce

$$-B \frac{\partial \varphi^*}{\partial t} - A\varphi^* = 0,\tag{6.7}$$

$$\varphi^* = \varphi_T^* \quad \text{at} \quad t = T\tag{6.8}$$

together with (6.4). For (6.7), (6.8), as can be easily seen the identity, analogous to (5.13), will take place, for the new five-dimensional phase space

$$(B\varphi_T, \varphi_T^*)_D = (B\varphi_0, \varphi_0^*)_D,\tag{6.9}$$

which can be presented in the explicit form

$$\begin{aligned}\int_D (\bar{\rho}u_T u_T^* + \bar{\rho}v_T v_T^* + \frac{g\bar{T}}{\gamma_a - \gamma} \bar{\rho}\vartheta_T \vartheta_T^*)dD = \\ \int_D (\bar{\rho}u_0 u_0^* + \bar{\rho}v_0 v_0^* + \frac{g\bar{T}}{\gamma_a - \gamma} \bar{\rho}\vartheta_0 \vartheta_0^*)dD.\end{aligned}\tag{6.10}$$

If we choose $u_T^* = u_T$, $v_T^* = v_T$, $\vartheta^* = \vartheta_T$ we come to the law of conservation total energy

$$\int_D \bar{\rho} \pi_T dD = \int_D \bar{\rho} \pi_0 dD.$$

where

$$\pi = u^2 + v^2 + \frac{g\bar{T}}{\gamma_a - \gamma} \vartheta^2.$$

Let the values u_T^* , v_T^* and ϑ_T^* be chosen as follows:

$$u_T^* = 0, v_T^* = 0, \vartheta_T^* = \frac{\gamma_a - \gamma}{g\bar{T}} \delta(x-x_0, y-y_0, z-z_0). \quad (6.11)$$

Then on the basis of (6.10) we obtain

$$\bar{\rho} \vartheta_T(x_0, y_0, z_0) = \int_D (\bar{\rho} u_0^* u_0^* + \bar{\rho} v_0^* v_0^* + \frac{g\bar{T}}{\gamma_a - \gamma} \bar{\rho} \vartheta_0^* \vartheta_0^*) dD. \quad (6.12)$$

This formula indicates connection between temperature in the given point of space at the time $t = T$ and the initial state of the atmosphere (at $t = 0$). It should be remembered that u_0 , v_0 and ϑ_0 in (6.12) are given at the initial time and u_0^* , v_0^* , ϑ_0^* are the solutions of conjugate equations under condition (6.11).

A BAROCLINIC MODEL ATMOSPHERE IN A NON-ADIABATIC APPROXIMATION

Let the following be a complete system of equations of the atmosphere's dynamics with regard to turbulent exchange and the given sources of heat

$$\frac{\partial \bar{\rho} u}{\partial t} + \Lambda u - l \bar{\rho} v + \bar{\rho} \frac{\partial \varphi}{\partial z} - \mu \bar{\rho} \Delta u = 0, \quad (7.1)$$

$$\begin{aligned} \frac{\partial \bar{\rho}v}{\partial t} + \Lambda v + l\bar{\rho}u + \bar{p} \frac{\partial \varphi}{\partial y} - \mu\bar{\rho}\Delta v &= 0, \\ -g\bar{\rho}\vartheta + \bar{p} \frac{\partial \varphi}{\partial z} &= 0, \end{aligned} \quad (7.1)$$

$$\frac{\partial \bar{\rho}u}{\partial x} + \frac{\partial \bar{\rho}v}{\partial y} + \frac{\partial \bar{\rho}w}{\partial z} = 0,$$

$$\frac{\partial \bar{\rho}\vartheta}{\partial t} + \Lambda\vartheta + \frac{\gamma_a - \gamma}{T} \bar{\rho}w - \frac{\partial}{\partial z} \nu_1 \bar{\rho} \frac{\partial \vartheta}{\partial z} - \mu_1 \bar{\rho} \Delta \vartheta = 0.$$

The boundary conditions are

$$\frac{\partial \vartheta}{\partial z} = \alpha_s (\vartheta - \bar{\vartheta}), \quad \bar{\rho}w = 0 \quad \text{at} \quad z = 0, \quad (7.2)$$

$$\frac{\partial \vartheta}{\partial z} = 0, \quad \bar{\rho}w = 0 \quad \text{at} \quad z = H.$$

Here the following notation is used:

α_s – coefficient of heat transfer assumed to be temporarily equal to zero on the ground and on the polar ice,

$\bar{\vartheta}$ – temperature of the surface layer of the ocean assumed to be known,

μ, μ_1 – turbulent exchange coefficients.

On the boundary of the domain in the plane (x, y) there is a condition of periodicity of the solution. Let

$$u = u_0, \quad v = v_0, \quad \vartheta = \vartheta_0 \quad \text{at} \quad t = 0 \quad (7.3)$$

be the initial data.

The solution is assumed to have absolutely continuous first order time derivatives with respect to u, v and ϑ and second order derivatives

in all space variables. Let us introduce into consideration the matrix operator

$$A = \begin{pmatrix} \Lambda - \mu\rho\Delta & -l\bar{\rho} & 0 & \bar{p} \frac{\partial}{\partial x} & 0 \\ l\bar{\rho} & \Lambda - \mu\bar{\rho}\Delta & 0 & \bar{p} \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & \bar{p} \frac{\partial}{\partial z} & -g\bar{\rho} \\ \frac{\partial}{\partial x} \bar{\rho} & \frac{\partial}{\partial y} \bar{\rho} & \frac{\partial}{\partial z} \bar{\rho} & 0 & 0 \\ 0 & 0 & g\bar{\rho} & 0 & \frac{g\bar{T}}{\gamma_a - \gamma} (\Lambda - \nu_1 \bar{\rho} \frac{\partial}{\partial z} - \mu_1 \bar{\rho} \Delta) \end{pmatrix}$$

and the vector

$$\varphi = \begin{pmatrix} u \\ v \\ w \\ \varphi \\ \vartheta \end{pmatrix}$$

Then we arrive at the problem

$$B \frac{\partial \varphi}{\partial t} + A\varphi = 0,$$

(7.4)

$$\varphi = \varphi_0 \quad \text{at} \quad t = 0.$$

Now we consider the conjugate operator A^*

$$= \begin{vmatrix} -\Lambda - \mu \bar{\rho} \Delta & l \bar{\rho} & 0 & -\bar{p} \frac{\partial}{\partial x} & 0 \\ -l \bar{\rho} & -\Lambda - \mu \bar{\rho} \Delta & 0 & -\bar{p} \frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 & -\bar{p} \frac{\partial}{\partial z} & g \bar{\rho} \\ -\frac{\partial}{\partial x} \bar{\rho} & -\frac{\partial}{\partial y} \bar{\rho} & -\frac{\partial}{\partial z} \bar{\rho} & 0 & 0 \\ 0 & 0 & -g \bar{\rho} & 0 & -\frac{g \bar{\Gamma}}{\gamma_a - \gamma} (-\Lambda - \frac{\partial}{\partial z} \nu_1 \frac{\partial}{\partial z} - \mu_1 \bar{\rho} \Delta) \end{vmatrix}$$

and formulate the following problem

$$-B \frac{\partial \varphi^*}{\partial t} + A^* \varphi^* = 0, \quad (7.5)$$

$$\varphi^* = \varphi_T^* \quad \text{at} \quad t = T.$$

Here the components of vector-function φ^* satisfy the requirements of smoothness, the conditions of periodicity on the boundary of the domain and the conditions

$$\begin{aligned} \frac{\partial \vartheta^*}{\partial z} = \alpha_s \vartheta^*, \quad \bar{\rho} w^* = 0 \quad \text{at} \quad z = 0, \\ \frac{\partial \vartheta^*}{\partial z} = 0, \quad \bar{\rho} w^* = 0 \quad \text{at} \quad z = H. \end{aligned} \quad (7.6)$$

Problem (7.5) in the component form is as follows:

$$\begin{aligned}
-\frac{\partial \bar{\rho} u^*}{\partial t} - \Lambda u^* + l \bar{\rho} v^* - \bar{p} \frac{\partial \varphi^*}{\partial x} - \mu \bar{\rho} \Delta u^* &= 0, \\
-\frac{\partial \bar{\rho} v^*}{\partial t} - \Lambda v^* - l \bar{\rho} u^* - \bar{p} \frac{\partial \varphi^*}{\partial y} - \mu \bar{\rho} \Delta v^* &= 0, \\
g \bar{\rho} \vartheta^* - \bar{p} \frac{\partial \varphi^*}{\partial z} &= 0,
\end{aligned} \tag{7.7}$$

$$\begin{aligned}
\frac{\partial \bar{\rho} u^*}{\partial x} + \frac{\partial \bar{\rho} v^*}{\partial y} + \frac{\partial \bar{\rho} w^*}{\partial z} &= 0, \\
-\frac{\partial \bar{\rho} \vartheta^*}{\partial t} - \Lambda \vartheta^* - \frac{\gamma_a - \gamma}{\bar{T}} \bar{\rho} w^* - \frac{\partial}{\partial z} \bar{\rho} \nu_1 \frac{\partial \vartheta^*}{\partial z} - \rho \mu_1 \Delta \vartheta^* &= 0
\end{aligned}$$

under the boundary conditions (7.6) and the initial data

$$u^* = u_T^*, v^* = v_T^*, \vartheta^* = \vartheta_T^* \quad \text{at} \quad t = T. \tag{7.8}$$

Since Λ and Λ^* are conjugate operators, the following conditions hold, at $\bar{\vartheta} = 0$:

$$(\Lambda \varphi, \varphi^*)_D = (\varphi, \Lambda^* \varphi^*)_D. \tag{7.9}$$

We scalarly multiply equations (7.4), (7.5) by φ^* and φ respectively, integrate them with time from 0 and T and subtract the results one from the other. Then, in a component form, we arrive at the expression

$$\begin{aligned}
\int_D (u_T u_T^* + v_T v_T^* + \frac{g \bar{T}}{\gamma_a - \gamma} \vartheta_T \vartheta_T^*) \bar{\rho} dD - \int_D (u_o u_o^* + v_o v_o^* + \\
\frac{g \bar{T}}{\gamma_a - \gamma} \vartheta_o \vartheta_o^*) \bar{\rho} dD - q \int_o^T dt \int_s \alpha_s \bar{\vartheta} \vartheta^* dS &= 0,
\end{aligned} \tag{7.10}$$

where $q = \frac{\nu_1 g \bar{\rho} \bar{T}}{\gamma_a - \gamma}$; S is the surface of the World Ocean with the given temperature of the surface layer $\bar{\vartheta}(x, y, 0, t)$.

Suppose that we are interested in the forecast of the mean temperature field in the domain $G \{ x, y \in \sigma, 0 \leq Z \leq h \}$. Then the "initial" conditions of the conjugate equations will be chosen as follows:

$$u_T^* = 0, v_T^* = 0,$$

$$\frac{g \bar{T}}{\gamma_a - \gamma} \vartheta_T^* = \frac{1}{G}, X \in G \text{ and } \vartheta_T^* = 0 \text{ outside } G, \quad (7.11)$$

where X is a set of coordinates (x, y, z) .

Introducing the following notation for the mean temperature anomaly in the domain G at the time $T = t$:

$$\frac{1}{G} \int_G \bar{\rho} \vartheta_T' dD = \bar{\rho} \vartheta_T^{-G}$$

we can write

$$\begin{aligned} \bar{\rho} \vartheta_T^{-G} = & \int_D (u_o u_o^* + v_o v_o^* + \frac{g \bar{T}}{\gamma_a - \gamma} \vartheta_o \vartheta_o^*) \bar{\rho} dD + \\ & q \int_0^T dt \int_s \alpha_s \bar{\vartheta} \vartheta^* dS. \end{aligned} \quad (7.12)$$

The expression $\bar{\rho} \vartheta_T^{-G}$ denotes that the mean temperature anomaly is calculated on the basis of the data on the interval $0 \leq t \leq T$.

Hence the problem of the forecast of the mean temperature anomaly reduces to the solution of the conjugate problem (7.7), (7.8) under condition (7.11).

In the present paragraph the theory of perturbations is constructed under specially set initial conditions for a system of conjugate equations and homogeneous boundary conditions. It will be shown that other formulations of problems are possible for a system of

conjugate equations which lead to the theory of perturbations of practical use. Indeed, let us consider a system of equations (7.1) with the boundary conditions (7.2) and the initial data (7.3) and then the system of conjugate equations (7.7).

The boundary conditions for system (7.7) are defined as follows

$$\begin{aligned} \frac{\partial \vartheta^*}{\partial z} &= \alpha_s \vartheta^* + f^*, \quad \bar{\rho} w^* = 0 \quad \text{at} \quad z = 0, \\ \frac{\partial \vartheta^*}{\partial z} &= 0, \quad \bar{\rho} w^* = 0 \quad \text{at} \quad z = H, \end{aligned} \quad (7.13)$$

where $f^*(x, y, z)$ has the form

$$f^* = \frac{\bar{\rho}}{G_o q} \delta(t - T), \quad \text{if} \quad (x, y) \in G_o,$$

$f^* = 0$ outside of the defined domain.

Here G_o is a region on the Earth for which a forecast of the mean temperature anomaly should be made.

The initial data for the conjugate problem are as follows:

$$u^* = 0, \quad v^* = 0, \quad \vartheta^* = 0 \quad \text{at} \quad t > T. \quad (7.14)$$

With the help of the basic and conjugate problems and the above techniques we obtain the functional

$$\begin{aligned} \bar{\rho} \vartheta_T^{-G_o} &= \int_T (u_o u_o^* + v_o v_o^* + \frac{g\bar{T}}{\gamma_a - \gamma} \vartheta_o \vartheta_o^*) \bar{\rho} dD + \\ & q \int_o^T dt \int_s \alpha_s \bar{\vartheta} \vartheta^* dS, \end{aligned} \quad (7.15)$$

where

$$\bar{\rho} \vartheta^{-G_0} = \frac{1}{G_0} \int_{G_0} \bar{\rho} \vartheta_T dS. \quad (7.16)$$

Assuming that we deal with small perturbations $u' = u + \delta u$, $v' = v + \delta v$, $\vartheta' = \vartheta + \delta \vartheta$, $\bar{\vartheta}' = \bar{\vartheta} + \delta \bar{\vartheta}$ and using the above methods we get

$$\begin{aligned} \delta(\bar{\rho} \vartheta_T^{-G_0}) = \int_D (\delta u_0 u_0^* + \delta v_0 v_0^* + \frac{g\bar{T}}{\gamma_a - \gamma} \delta \vartheta_0 \vartheta_0^*) \bar{\rho} dD + \\ q \int_0^T dt \int_s \alpha_s \delta \bar{\vartheta} \vartheta^* dS. \end{aligned} \quad (7.17)$$

Comparing (7.15) and (7.12) we can see that they coincide if in (7.12) we choose $G \equiv G_0$. It is easy to see that in this case the solutions of conjugate equations will also coincide because it does not matter whether the instantaneous "source" on G_0 is defined in the boundary conditions or as the Cauchy condition at the moment $t = T$. Up to now we have been assuming that our task is to solve a long-term weather forecast for the moment $t = T$ with respect to the initial $t = 0$. In making forecasts for periods of one month or a season there is no sense to define exactly the moment of a forecast of temperature anomalies and other elements. From the methodical point of view it would be more correct to give a forecast of the mean temperature anomaly for a certain period of time. For example, in a weather forecast for the nearest month it would be reasonable to give ten-day averaged forecasts for every ten days of the month. If we are interested in a weather forecast for a period of one season, it would be enough to give averaged forecasts for the first, second and third months of the season.

Since the averaging of forecasts results in additional filtration of meteorological noise such procedure would increase informative significance of forecasts. Therefore, it is necessary to change a little our

approach to derivation of the formulas of the theory of perturbations.

For this purpose for a system of conjugate equations we take the following boundary conditions:

$$\begin{aligned} \frac{\partial \vartheta^*}{\partial z} &= \alpha_s \vartheta^* + f^*, \quad \bar{\rho} w^* = 0 \quad \text{at} \quad z = 0, \\ \frac{\partial \vartheta^*}{\partial z} &= 0, \quad \bar{\rho} w^* = 0 \quad \text{at} \quad z = H, \end{aligned} \quad (7.18)$$

where $f^*(x, y, t)$ is given in the form:

$$f^*(x, y, t) = \begin{cases} \bar{\rho} \xi^*(x, y) \eta^*(t), & \text{if } (x, y) \in G_0, t \in [T - \tau, T], \\ \bar{g} & \\ 0 & \text{outside the domain} \end{cases}$$

The functions $\xi^*(x, y)$ and $\eta^*(t)$ are assumed to be normed i.e.

$$\int_{G_0} \xi^*(x, y) dS = 1, \quad \int_{T-\tau}^T \eta^*(t) dt = 1.$$

In a special case, naturally, they can be constant.

The initial data for conjugate equations are taken as follows:

$$u_T^* = 0, \quad v_T^* = 0, \quad \vartheta_T^* = 0 \quad \text{at} \quad t > T. \quad (7.19)$$

Applying the boundary condition (7.18) and the initial data (7.19) we obtain, by the usual method, the functional

$$\bar{\rho} \vartheta_{T-\frac{\tau}{2}}^{-G_0} = \int_D (u_0 u_0^* + v_0 v_0^* + \frac{g\bar{T}}{\gamma_a - \gamma} \vartheta_0 \vartheta_0^*) \bar{\rho} dD + \quad (7.20)$$

$$q \int_0^T dt \int_s \alpha_s \bar{\vartheta}^* dS, \quad (7.20)$$

where

$$\bar{\rho} \vartheta_{T - \frac{\tau}{2}}^{-G_0} = \int_{T - \tau}^T \eta^*(t) dt \int_{G_0} \bar{\rho} \vartheta \xi^*(x, y) dS. \quad (7.21)$$

Comparing formula (7.20) with (7.15) and (7.12) one can see that they coincide, only the meaning of the conjugate solution changes. In practice it is convenient to choose ξ^* and η^* as smooth and positive functions. In this case expression (7.21) has the meaning of the temperature value averaged with weight in the defined region G_0 and the time interval $t - \tau \leq t \leq T$.

THE VALUE OF METEOROLOGICAL INFORMATION WITH RESPECT TO THE MEAN TEMPERATURE ANOMALIES AND EXTENDED-RANGE FORECAST

The perturbation formulas derived above allow us to give interpretation of the conjugate problem as the value of information with respect to the temperature anomaly in a defined domain. To explain this, it is necessary to describe, at least qualitatively, the dynamics of the solution of the conjugate problem

$$- B \frac{\partial \varphi^*}{\partial t} + A^* \varphi^* = 0, \quad (8.1)$$

$$B\varphi^* = B\varphi_T^* \quad \text{at} \quad t = T,$$

where conditions (7.11), for example, are chosen as components of vector $B\varphi_T^*$. It is not difficult to conceive that if $t = T$, according to (7.11) we have $u_T^* = 0$, $v_T^* = 0$ and ϑ_T^* is different

from zero only in the domain G where it is a constant. We solve the system of conjugate equations for $t < T$ assuming that functions u , v , w are known. In this case, due to the transfer of substances, the domain of non-zero initial data will move westward approximately as far as $u\Delta t$, north (or south) -ward as far as $v\Delta t$, and along z as far as $w\Delta t$, where $\Delta t = T - t$. This, certainly, immediately causes gravitational (Rossby) waves which "smear" the picture extending the domain of perturbation and so on. Due to turbulent diffusion the intensity of the fields of the components of conjugate functions will be constantly falling tending to zero within $t \rightarrow -\infty$. This means that for sufficiently remote past times with respect to the moment $t = T$, due to dissipative processes, the information about the initial fields will be useless as it turns into meteorological noise. In the mathematical model this is reflected as follows: the first integral of formula (7.12) tends to zero at $t \rightarrow -\infty$, and variations of the temperature anomaly are determined only by the heat flux from the ocean, i.e.

$$\delta (\bar{\rho}^G_T) = q \int_{-\infty}^T dt \int_S \alpha_s \delta \bar{\vartheta} \vartheta^* dS. \quad (8.2)$$

This limiting relation allows us to make a very important conclusion about the role of the ocean in forming temperature anomalies in extended-range weather forecast. Moreover, the construction of formula (8.2) shows that conjugate function ϑ^* (solution of the conjugate problem) is influence function with respect to the variations being forecasted.

Since the conjugate solution is, eventually, the fundamental criterion of the significance of information with respect to the functional of the problem, it can be called the value of information.

THE GENERAL PERTURBATION THEORY FOR EVOLUTIONARY PROBLEMS

In the above sections of this paper the perturbation theory was constructed on the assumption that the actual field of velocity vector differs but slightly from the climatic one. In fact we deal with rather essential deviations of the fields of meteorological elements from the climatic ones especially in the short-term weather forecasting. In this case the theory of small perturbations appears to be inadequate. Thus we come to the necessity to develop a more complete theory which is not associated with the assumption of small perturbations. To this aim we consider the climatic state of the atmosphere as unperturbed and the actual state of the atmosphere as perturbation. Let the unperturbed state of the atmosphere be described by problem (7.4).

$$B \frac{\partial \varphi}{\partial t} + A\varphi = f,$$

$$B\varphi = B\varphi_0 \quad \text{at} \quad t = 0 \quad (9.1)$$

and the conjugate problem (8.1)

$$-B \frac{\partial \varphi^*}{\partial t} + A^*\varphi^* = 0, \quad (9.2)$$

$$B\varphi^* = B\varphi_T^* \quad \text{at} \quad t = T$$

In the phase space $D \times T$ we introduce scalar product

$$(g, h)_{D \times T} = \sum_{i=1}^5 \int_0^T dt \int_D g_i h_i dD. \quad (9.3)$$

Then we multiply scalarly equation (9.1) by φ^* and equation (5.2) by φ and subtract the results. Making use of the initial data and

conditions for the components φ and φ^* on the boundaries of the domain D^* we obtain

$$(B\varphi_T, \varphi_T^*)_D - (B\varphi_0, \varphi_0^*)_D + \int_0^T dt [(\varphi^*, A\varphi)_D - (\varphi, A^*\varphi^*)_D] = \int_0^T (f, \varphi^*)_D dt \quad (9.4)$$

Taking (7.9) into account,

$$\int_0^T [(\varphi^*, A\varphi)_D - (\varphi, A^*\varphi^*)_D] dt = 0. \quad (9.5)$$

Relation (9.4) becomes

$$(B\varphi_T, \varphi_T^*)_D - (B\varphi_0, \varphi_0^*)_D = (f, \varphi^*)_D. \quad (9.6)$$

Thus operators of the solution and the input data for the initial and conjugate equations in the unperturbed state are related by dependences (9.4) – (9.6).

Along with the unperturbed state, let us consider the perturbed state of the atmosphere. Let it be described by

$$B \frac{\partial \varphi'}{\partial t} + A' \varphi' = f',$$

$$B\varphi' = B\varphi'_0 \quad \text{at} \quad t = 0. \quad (9.7)$$

Let the conjugate problem corresponding to the unperturbed state of the atmosphere be joined to (9.7)

$$-B \frac{\partial \varphi^*}{\partial t} + A^* \varphi^* = 0, \quad (9.8)$$

$$B\varphi^* = B\varphi_T^* \quad \text{at} \quad t = T$$

Multiplying equation (9.7) scalarly by φ^* , equation (9.8) – by φ' and subtracting the results, we get a relation similar to (9.4)

$$(B\varphi'_T, \varphi^*_T)_D - (B\varphi'_0, \varphi^*_0)_D + \int_0^T dt [(\varphi^*, A'\varphi')_D - (\varphi', A^*\varphi^*)_D] = \int_0^T (f', \varphi^*)_D dt + R, \quad (9.9)$$

where R is some functional denoting that the boundary conditions for the components of the solution φ can be non-homogeneous. The form of this functional for some particular cases will be shown later. Let

$$A' = A + \delta A, \quad \varphi' = \varphi + \delta\varphi, \quad f' = f + \delta f, \quad (9.10)$$

where A, φ and f are the operator and vectors corresponding to the unperturbed state.

Substituting (9.10) into (9.9) and making use of (9.4) and the fact that

$$\int_0^T dt(\varphi^*, A\varphi')_D = \int_0^T dt(\varphi', A^*\varphi^*)_D + R \quad (9.11)$$

we obtain the formula of perturbations in the form

$$(B\delta\varphi_T, \varphi^*_T)_D - (B\delta\varphi_0, \varphi^*_0)_D + \int_0^T (\varphi^*, \delta A\varphi')_D dt = \int_0^T (\delta f, \varphi^*)_D dt + \delta R. \quad (9.12)$$

Formula (9.12) is the main one for obtaining different prognostic expressions for the functionals sought for.

THE PERTURBATION THEORY FOR THE PROBLEMS OF WEATHER FORECASTING

Let us turn to the component representation of the perturbation theory formulas. To this aim we consider the perturbed system of equations of atmospheric motions

$$\begin{aligned} \frac{\partial \bar{\rho} u'}{\partial t} + \Lambda' u' - l \bar{\rho} v' - \bar{p} \frac{\partial \varphi'}{\partial x} - \mu \bar{\rho} \Delta u' &= 0, \\ \frac{\partial \bar{\rho} v'}{\partial t} + \Lambda' v' - l \bar{\rho} u' - \bar{p} \frac{\partial \varphi'}{\partial y} - \mu \bar{\rho} \Delta v' &= 0, \\ g \bar{\rho} \vartheta' - \bar{p} \frac{\partial \varphi'}{\partial z} &= 0, \end{aligned} \quad (10.1)$$

$$\frac{\partial \bar{\rho} u'}{\partial x} + \frac{\partial \bar{\rho} v'}{\partial y} + \frac{\partial \bar{\rho} w'}{\partial z} = 0,$$

$$\frac{\partial \bar{\rho} \vartheta'}{\partial t} + \Lambda' \vartheta' + \frac{\gamma_a - \gamma}{\bar{T}} \bar{\rho} w' - \frac{\partial}{\partial z} \bar{\rho} \nu_1 \frac{\partial \vartheta'}{\partial z} - \bar{\rho} \mu_1 \Delta \vartheta' = 0$$

with the boundary conditions

$$\begin{aligned} \frac{\partial \vartheta'}{\partial z} &= \alpha'_s (\vartheta' - \bar{\vartheta}'), \quad \bar{\rho} w' = 0 \quad \text{at} \quad z = 0, \\ \frac{\partial \vartheta'}{\partial z} &= 0, \quad \bar{\rho} w' = 0 \quad \text{at} \quad z = H \end{aligned} \quad (10.2)$$

and the conditions of periodicity with respect to (x, y) . Here $\vartheta' = \vartheta + \delta \vartheta$, $\bar{\vartheta}' = \bar{\vartheta} + \delta \bar{\vartheta}$, ϑ and $\bar{\vartheta}$ are climatic surface air temperatures and

the temperature of the upper friction layer of the ocean, respectively, $\delta\vartheta$ and $\delta\bar{\vartheta}$ are deviations from climatic values. We take

$$u' = u'_0, v' = v'_0, \vartheta' = \vartheta'_0 \quad \text{at} \quad t = 0. \quad (10.3)$$

as initial data.

Now the conjugate system corresponding to the unperturbed state of the atmosphere is considered

$$\begin{aligned} -\frac{\partial \bar{\rho} u^*}{\partial t} - \Lambda u^* + l \bar{\rho} v^* - \bar{p} \frac{\partial \varphi^*}{\partial z} - \mu \bar{\rho} \Delta u^* &= 0, \\ -\frac{\partial \bar{\rho} v^*}{\partial t} - \Lambda v^* - l \bar{\rho} u^* - \bar{p} \frac{\partial \varphi^*}{\partial y} - \mu \bar{\rho} \Delta v^* &= 0, \\ g \bar{\rho} \vartheta^* - \bar{p} \frac{\partial \varphi^*}{\partial z} &= 0, \end{aligned} \quad (10.4)$$

$$\frac{\partial \bar{\rho} u^*}{\partial x} + \frac{\partial \bar{\rho} v^*}{\partial y} + \frac{\partial \bar{\rho} w^*}{\partial z} = 0,$$

$$-\frac{\partial \bar{\rho} \vartheta^*}{\partial t} - \Lambda \vartheta^* - \frac{\gamma_a - \gamma}{\bar{T}} \bar{\rho} w^* - \frac{\partial_l}{\partial z} \bar{\rho} v_1 \frac{\partial \vartheta^*}{\partial z} - \bar{\rho} \mu_1 \Delta \vartheta^* = 0$$

with the boundary conditions

$$\begin{aligned} \frac{\partial \vartheta^*}{\partial z} &= \alpha_s \vartheta^*, \quad \bar{\rho} w^* = 0 \quad \text{at} \quad z = 0, \\ \frac{\partial \vartheta^*}{\partial z} &= 0, \quad \bar{\rho} w^* = 0 \quad \text{at} \quad z = H \end{aligned} \quad (10.5)$$

assuming the periodicity of solution with respect to (x, y) and the initial data

$$u^* = u_T^*, v^* = v_T^*, \vartheta^* = \vartheta_T^* \quad \text{at} \quad t = T, \quad (10.6)$$

where u_T^* , v_T^* and ϑ_T^* are functions to be defined below. Before constructing the formulas of the perturbation theory let us introduce the following notation:

$$\Lambda' = \Lambda + \delta\Lambda, \quad \alpha'_s = \alpha_s + \delta\alpha_s.$$

Repeating the above mentioned procedure we multiply the initial equation of system (10.1) by u^* , v^* , w^* , $RT\varphi^*$ and $\frac{g\bar{\Gamma}}{\gamma_a - \gamma} \vartheta^*$ respectively, and then sum up the results. In a similar way we multiply the equations of the conjugate system (10.4) by u , v , w' , $RT\varphi$ and $\frac{g\bar{\Gamma}}{\gamma_a - \gamma} \vartheta'$ respectively, and sum up the results. From the expression obtained for system (10.1) we subtract the one obtained for system (10.4). Integrating the result over the domain $D \times T$ and making simple transformation we get

$$\int_D (u'_T u_T^* + v'_T v_T^* + \frac{g\bar{\Gamma}}{\gamma_a - \gamma} \vartheta'_T \vartheta_T^*) \bar{\rho} \, dD - \int_D (u'_0 u_0^* + v'_0 v_0^* + \frac{g\bar{\Gamma}}{\gamma_a - \gamma} \vartheta'_0 \vartheta_0^*) \bar{\rho} \, dD + \int_0^T dt \int_D (u^* \delta\Lambda u' + v^* \delta\Lambda v' + \frac{g\bar{\Gamma}}{\gamma_a - \gamma} \vartheta^* \delta\Lambda \vartheta') \, dD - q \int_0^T dt \int_S (\alpha_s \bar{\vartheta}' - \delta\alpha_s \vartheta') \vartheta^* \, dS = 0. \quad (10.7)$$

Assuming that perturbations of the operators and of solutions are absent we obtain

$$\int_D (u_T u_T^* + v_T v_T^* + \frac{g\bar{\Gamma}}{\gamma_a - \gamma} \vartheta_T \vartheta_T^*) \bar{\rho} dD - \int_D (u_o u_o^* + v_o v_o^* + \frac{g\bar{\Gamma}}{\gamma_a - \gamma} \vartheta_o \vartheta_o^*) \bar{\rho} dD - g \int_o^T dt \int_s \alpha_s \bar{\vartheta} \vartheta^* dS = 0. \quad (10.8)$$

Using the notation

$$u' = u + \delta u, v' = v + \delta v, \vartheta' = \vartheta + \delta \vartheta$$

and subtracting (10.8) from (10.7) we come to

$$\begin{aligned} & \int_D (\delta u_T \cdot u_T^* + \delta v_T \cdot v_T^* + \frac{g\bar{\Gamma}}{\gamma_a - \gamma} \delta \vartheta_T \cdot \vartheta_T^*) \bar{\rho} dD - \\ & \int_D (\delta u_o \cdot u_o^* + \delta v_o \cdot v_o^* + \frac{g\bar{\Gamma}}{\gamma_a - \gamma} \delta \vartheta_o \cdot \vartheta_o^*) \bar{\rho} dD + \int_o^T dt \int_D (u^* \delta \Lambda u' + v^* \delta \Lambda v' + \\ & \frac{g\bar{\Gamma}}{\gamma_a - \gamma} \vartheta^* \delta \Lambda \vartheta') \bar{\rho} dD - q \int_o^T dt \int_s [\alpha_s \delta \bar{\vartheta} + \\ & \delta \alpha_s (\vartheta' - \vartheta')] \vartheta^* dS = 0. \end{aligned} \quad (10.9)$$

The final form of the formula of the perturbation theory will be

$$\int_D (\delta u_T \cdot u_T^* + \delta v_T \cdot v_T^* + \frac{g\bar{\Gamma}}{\gamma_a - \gamma} \delta \vartheta_T \cdot \vartheta_T^*) \bar{\rho} dD = \quad (10.10)$$

$$\int_D (\delta u_o \cdot u_o^* + \delta v_o \cdot v_o^* + \quad (10.10)$$

$$\frac{g\bar{T}}{\gamma_a - \gamma} \delta \vartheta_o \cdot \vartheta_o^*) \bar{\rho} dD - \int_0^T dt \int_D (u^* \delta \Lambda u' + v^* \delta \Lambda v' +$$

$$\frac{g\bar{T}}{\gamma_a - \gamma} \vartheta^* \delta \Lambda \vartheta') \bar{\rho} dD + q \int_0^T dt \int_s [\alpha_s \delta \bar{\vartheta} + \delta \alpha_s (\bar{\vartheta}' - \vartheta')] \cdot \vartheta^* dS.$$

If we take (7.11) as the initial conditions for the conjugate equations, we arrive at the formula for mean temperature anomalies

$$\delta(\bar{\rho} \vartheta_T^{-G}) = \int_D (\delta u_o \cdot u_o^* + \delta v_o \cdot v_o^* + \frac{g\bar{T}}{\gamma_a - \gamma} \delta \vartheta_o \cdot \vartheta_o^*) \bar{\rho} dD -$$

$$\int_0^T dt \int_D (u^* \delta \Lambda u' + v^* \delta \Lambda v' + \frac{g\bar{T}}{\gamma_a - \gamma} \vartheta^* \delta \Lambda \vartheta') \bar{\rho} dD +$$

$$q \int_0^T dt \int_s [\alpha_s \delta \bar{\vartheta} + \delta \alpha_s (\bar{\vartheta}' - \vartheta')] \vartheta^* dS. \quad (10.11)$$

Let us simplify (10.11) for the case of sufficiently long term of forecasting (a month, for example). Here, as mentioned above, the effect of the initial data is small since intensity of the values u^* , v^* and ϑ^* due to dissipation diminishes with time. Then formula (10.11) has the form

$$\delta(\bar{\rho} \vartheta_T^{-G}) = - \int_{-\infty}^T dt \int_D (u^* \delta \Lambda u' + v^* \delta \Lambda v' + \frac{g\bar{T}}{\gamma_a - \gamma} \vartheta^* \delta \Lambda \vartheta') dD +$$

$$q \int_{-\infty}^T dt \int_s [\alpha_s \delta \bar{\vartheta} + \delta \alpha_s (\bar{\vartheta}' - \vartheta')] \vartheta^* dS. \quad (10.12)$$

Assuming that the conjugate problem is solved at actual values of u, v, w then $\delta\Lambda = 0$ and we obtain

$$\delta(\bar{\rho}\vartheta_T^{-G}) = q \int_{-\infty}^T dt \int_s [\alpha_s \delta\bar{\vartheta} + \delta\alpha_s (\bar{\vartheta}' - \vartheta')] \vartheta^* dS. \quad (10.13)$$

The meaning of this formula is quite clear –the first term on the right (10.13) describes contribution into the temperature anomaly at the expense of the atmosphere– ocean interaction where

$$q \int_{-\infty}^T dt \int_s \alpha_s \delta\bar{\vartheta} \vartheta^* dS$$

makes allowance for the deviation of temperature of the surface friction layer from the climatic temperature, and the other term

$$q \int_{-\infty}^T dt \int_s \delta\alpha_s (\bar{\vartheta}' - \vartheta') \vartheta^* dS$$

describes the effects of deviation of the atmosphere-ocean heat transfer due to storms, non-standard dynamics of the ice cover, etc. It is evident from (10.14) that long-term temperature anomalies of large regions of the continents develop in the active layer of the ocean due to its interaction with the atmosphere.

Finally, if storms are neglected and the dynamics of ice is assumed known, the formula for forecasts of temperature anomalies has the simple form

$$\delta(\bar{\rho}\vartheta_T^{-G}) = q \int_{-\infty}^T dt \int_s \alpha_s \delta\bar{\vartheta} \vartheta^* dS. \quad (10.14)$$

In conclusion it should be noted that above we have considered the case when the sources in conjugate equations are presented in the

initial data (7.11). All these arguments hold true for a case when the sources are given in the boundary conditions in the form of (7.13), only G should be substituted then by G_0 and the solutions of problems (7.7), (7.13), (7.14) should be taken as conjugate functions.

APPENDIX

As an example we will analyze a simple problem. Given the values of u , v , w and cloudiness we solve the problem on thermal ocean-atmosphere (and continents) interaction taking account of the heat advection and convection, turbulent (horizontal and vertical) mixing. The problem is solved for deviations of temperature from the climatic value on the basis of information on climate for the temperature fields in the atmosphere, ocean and ground, cloudiness and velocity components of air and water particles at given levels. In addition it is necessary to have actual daily data on these meteorological elements in the atmosphere. However, daily data about the ground and the ocean are not used by the present model.

Following is a mathematical formulation of the problem. We solve the equation

$$\bar{\rho} \frac{dT}{dt} = \frac{\partial}{\partial z} \bar{\rho} \nu \frac{\partial T}{\partial z} + \bar{\rho} \mu \Delta T + \frac{\delta F}{C_p} \frac{\delta(z)}{\delta z}, \quad (1)$$

where T is deviation of temperature from the climatic value

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} + \bar{v} \frac{\partial}{\partial y} + \bar{w} \frac{\partial}{\partial z},$$

\bar{u} , \bar{v} , \bar{w} are climatic values of the velocity vector components. The boundary conditions are as follows:

for the atmosphere-ocean system

$$\bar{\rho} \nu \frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = H,$$

$$\bar{\rho} \nu \frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = -h_s \quad (2a)$$

for the atmosphere-continent system

$$\bar{\rho} \nu \frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = H,$$

$$\bar{\rho} \nu \frac{\partial T}{\partial z} = 0 \quad \text{at} \quad z = -h_c. \quad (2b)$$

The initial conditions are

$$T = T_0 \quad \text{at} \quad t = 0. \quad (3)$$

Here \bar{u} , \bar{v} and \bar{w} are assumed to be known values.

The initial problem (1, 2a, 2b, 3) is set to correspond with the conjugate problem

$$-\bar{\rho} \frac{dT^*}{dt} = \frac{\partial}{\partial z} \bar{\rho} \nu \frac{\partial T^*}{\partial z} + \bar{\rho} \mu \Delta T^* + \frac{F^* \delta(z)}{C_p}, \quad (4)$$

where T^* is conjugate temperature and the operator $\frac{d}{dt}$ is the same as in the initial problem, i.e. its coefficients are climatic values.

It should be noted that the value $\delta F(x, y, z, t)$ is dependent on deviations of cloudiness and the temperature of the oceanic surface layer from the climatic value. F^* is set depending on the nature of the functional.

When it is necessary to define the temperature anomaly averaged with respect to space and some time interval τ the function F^* is found as follows:

$$F^* = \begin{cases} \frac{1}{\tau \Delta G}, & \text{if } (x, y, z) \in G \text{ and simultaneously, } t \in \tau \\ 0 & \text{outside this interval} \end{cases} \quad (5)$$

The boundary conditions for equation (4) are:
for the atmosphere-ocean system

$$\begin{aligned} \bar{\rho} \nu \frac{\partial T^*}{\partial z} &= 0 \quad \text{at } z = H, \\ \bar{\rho} \nu \frac{\partial T^*}{\partial z} &= 0 \quad \text{at } z = -h_s \end{aligned} \quad (6a)$$

for the atmosphere-continent system

$$\begin{aligned} \bar{\rho} \nu \frac{\partial T^*}{\partial z} &= 0 \quad \text{at } z = H, \\ \bar{\rho} \nu \frac{\partial T^*}{\partial z} &= 0 \quad \text{at } z = -n_c. \end{aligned} \quad (6b)$$

The initial condition is

$$T^* = T_o^* \quad \text{at } t = t_o + \frac{\tau}{2}, \quad (7)$$

where t_o is the time for which the prediction is given. For the case when T^* is defined the temperature anomaly, averaged with respect to the domain G and the interval τ , is found by the formula

$$\bar{\delta T}_{\tau}^G = \int_{-\infty}^{t_o + \frac{\tau}{2}} dt \iint_{S+C} \delta F \cdot T^* dS. \quad (8)$$

In Figs. 1 to 6 isolines of the function T^* are shown at the level of the Earth's surface, the function T^* is averaged in time for a seven-day period. Fig. 1 refers to the first week of calculation, Figs. 2, 3, 4, 5 and 6 -to the last weeks of the first, second, third and fifth months. The territory being predicted is the European part of the USSR, the function T^* is calculated for a week-long prediction.

In this formulation the conjugate solution is the influence function with respect to the variations of the functional being predicted and is the principal criterion of significance of meteorological information.

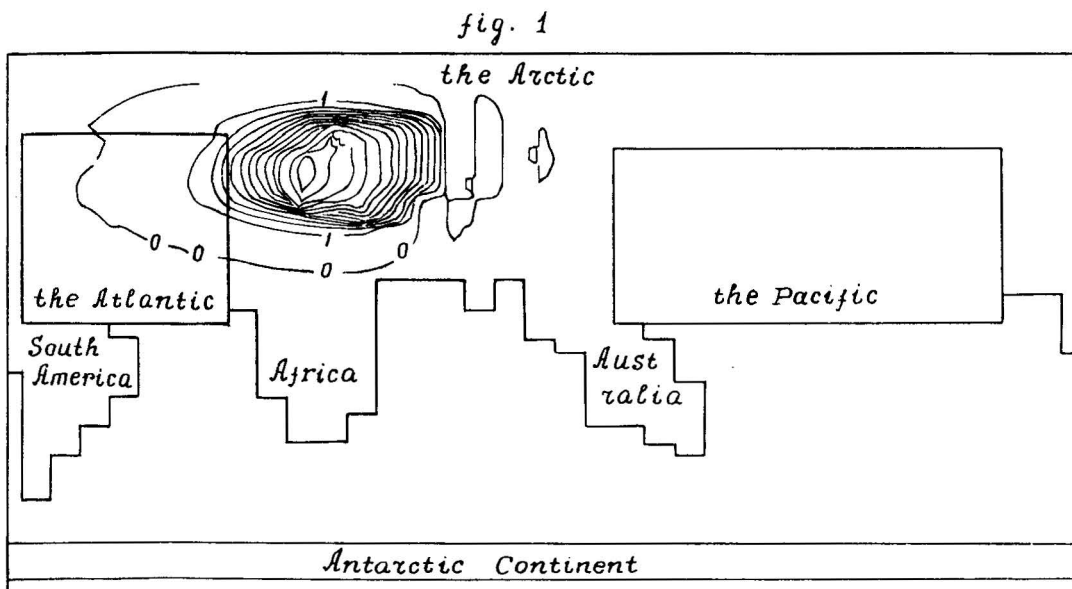


fig. 2.

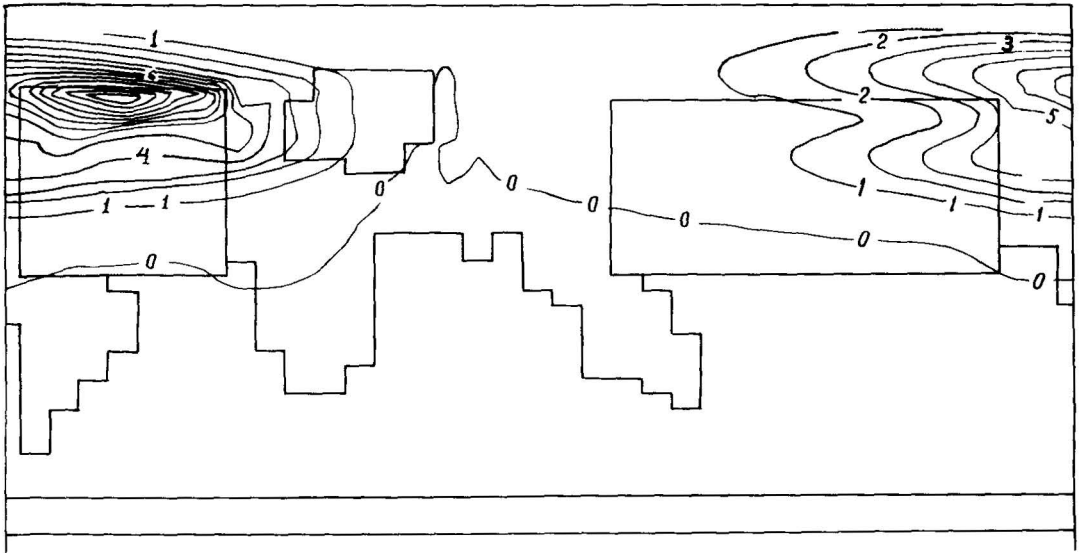


fig. 3.

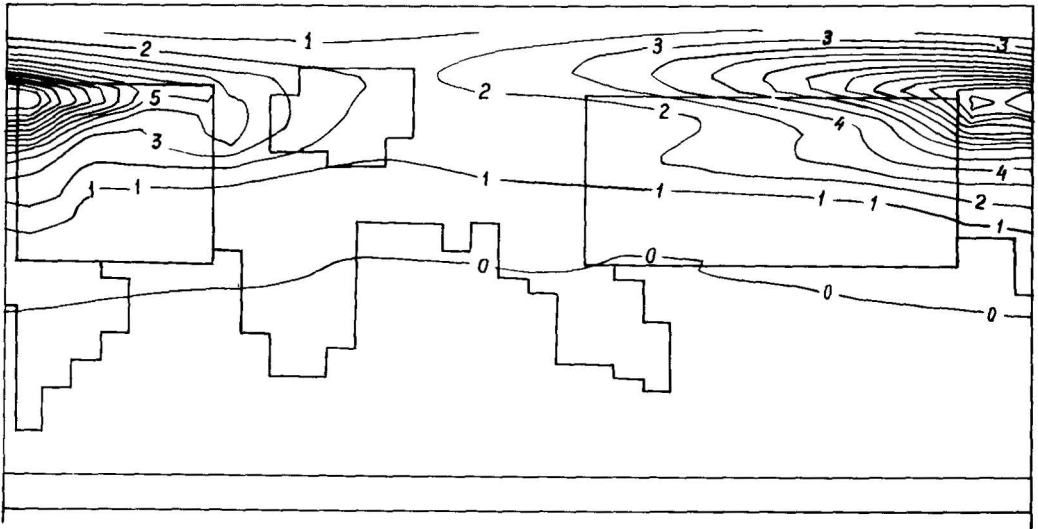


fig. 4.

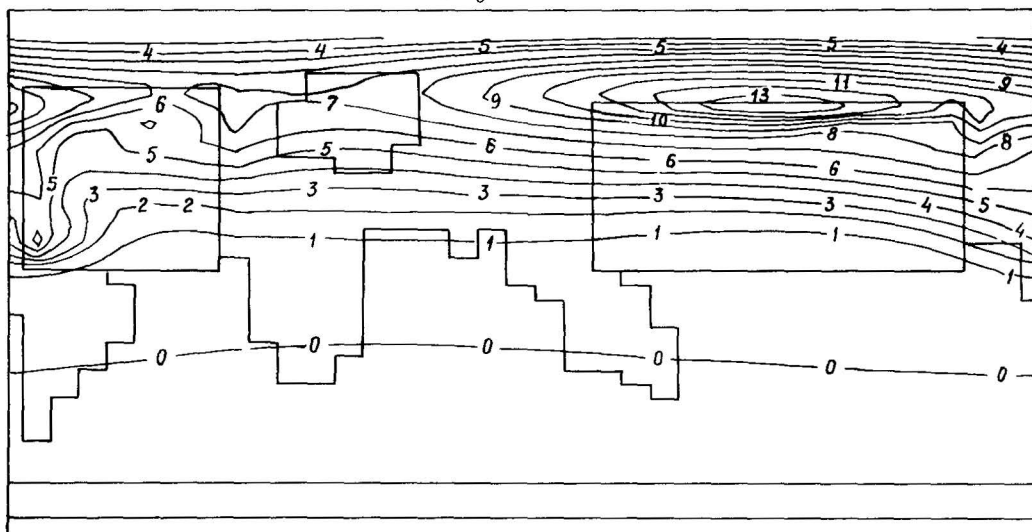


fig. 5.

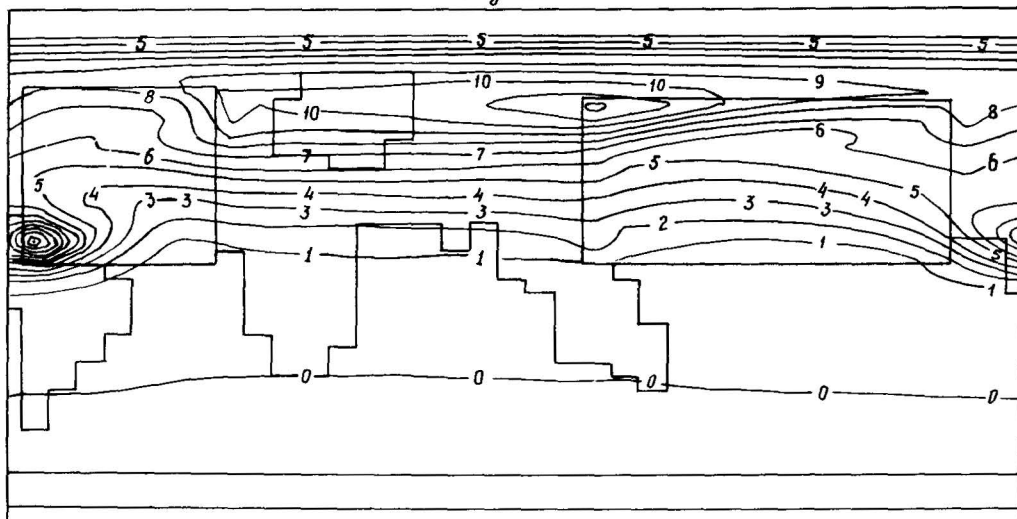
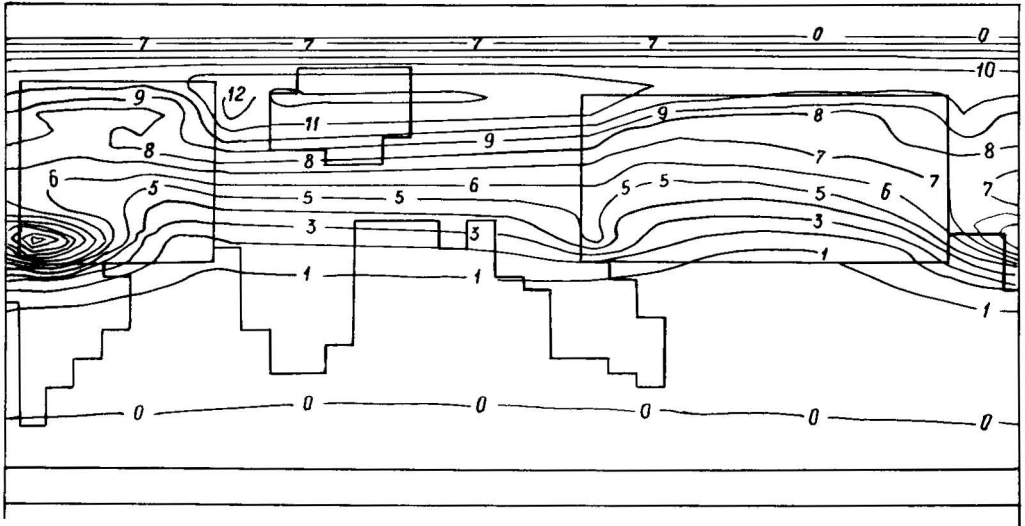


fig. 6.



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