# A NOTE ON DEEP KELVIN WAVES AND THEIR PROPAGATION THROUGH A SHEAR FLOW*

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## RESUMEN

Los actuales experimentos de laboratorio sobre los efectos topográficos de fluidos rotantes no homogéneos presentan ciertas interrogantes teóricas que conciernen a las propiedades de las ondas en un fluido profundo y estratificado en rotación. Las ecuaciones gobernantes admiten una clase de soluciones correspondientes a las "ondas inerciales-gravitacionales" que ocupan todo el fluido y otra clase que corresponde a las "ondas de Kelvin" que están más localizadas. Las propiedades características de las ondas de Kelvin son: (a) trayectorias lineales y co-planares de elementos fluidos individuales sin movimiento, en una dirección horizontal, (b) sentido de propagación de fase dependiente *inter alia* del signo del parámetro de Coriolis f, (c) velocidad de propagación de fase independiente de f, y (d) amplitud decayendo exponencialmente en la dirección de movimiento no horizontal a una razón proporcional a f. Las "ondas profundas Kelvin" corresponden al caso cuando el número de onda horizontal no es pequeño comparado con el número de onda vertical.

"El nivel crítico" bajo el cual las ondas de Kelvin en un flujo cizallante, estarán en su mayoría confinadas, está localizado donde la frecuencia intrínseca de la onda (i.e. la frecuencia Doppler relativa al flujo local) es igual a cero, como ocurre en ondas gravitacionales ordinarias no afectadas por la rotación, y no donde la frecuencia intrínseca es igual a $\pm$ f que es el caso de las ondas inerciales-gravitacionales.

## ABSTRACT

Current laboratory experiments on topographic effects in rotating non-homogeneous fluids pose certain theoretical questions concerning the properties of waves in a deep rotating stratified fluid. The governing equations admit one class of solutions corresponding to "inertia-gravity waves", which occupy the whole fluid, and another to "Kelvin waves", which are more localized. Characteristic properties of Kelvin waves are (a) linear and co-planar trajectories of individual fluid elements, with no motion in one horizontal direction, (b) sense of phase propagation dependent *inter alia* on the sign of the Coriolis parameter f, (c) speed of phase propagation independent of f and (d) amplitude decaying exponentially in the direction of no horizontal motion at a rate proportional to f. "Deep Kelvin waves" correspond to the case when the horizontal wavenumber is not small compared with the vertical wavenumber.

The "critical level" below which Kelvin waves in a shear flow will be largely confined is located where the intrinsic frequency of the wave (i.e. the Doppler frequency relative to the local flow) is equal to zero, as for ordinary gravity waves unaffected by rotation, and *not* where the intrinsic frequency is equal to $\pm$ f which is the case for inertia-gravity waves.


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## INTRODUCTION

Current laboratory experiments on topographic effects in rotating non-homogeneous fluids (the results of which will be reported elsewhere) pose certain theoretical questions concerning the properties of waves in a deep rotating stratified fluid. The governing hydrodynamical equations admit one class of solutions corresponding to "inertia-gravity waves", which occupy the whole fluid, and another to "Kelvin waves", which are more localized (see e.g. Huppert and Stern, 1974). Characteristic properties of Kelvin waves are: (a) linear and co-planar trajectories of individual fluid elements, with no motion in one horizontal direction, (b) sense of phase propagation dependent *inter alia* on the sign of the Coriolis parameter f, (c) speed of phase propagation independent of f, and (d) amplitude decaying exponentially in the direction of no horizontal motion at a rate proportional to f. "Deep Kelvin waves" simply correspond to the case when the horizontal wavenumber is not small in comparison with the vertical wavenumber.

The purpose of this note is to discuss the structure of Kelvin waves in a shear flow, one finding of particular interest being that the "critical level" below which the wave will be largely confined is located where the intrinsic frequency of the wave (i.e. the Doppler-shifted frequency relative to the local flow) is equal to zero, as for the internal gravity wave unaffected by rotation (Booker and Bretherton, 1967) and not where the intrinsic frequency is equal to $\pm$ f, which is the case for inertia-gravity waves (Jones, 1967).

## INERTIA-GRAVITY WAVES

Consider the motion of an inviscid and thermally-insulating "Boussinesq" fluid of undisturbed density $\rho_0$ and buoyancy frequency N, rotating about a vertical axis with angular velocity f/2. The position vector $\mathbf{r}$ = (x, y, z) and acceleration due to gravity $\mathbf{g}$ = (o, o,-g). Writing velocity, density and pressure as

$$u' = (u_o, o, o) + (u, v, w)$$

$$\rho' = \rho_o + \rho \tag{1}$$

and $$p' = p_o + p$$

respectively, where (u, v, w), $\rho$ and p are small perturbations about the mean state, the linearized equations governing the system are the momentum equations

$$\left(\frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x}\right) u + fv = - \frac{1}{\rho_o} \frac{\partial p}{\partial x}, \tag{2}$$

$$\left(\frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x}\right) v - fu = - \frac{1}{\rho_o} \frac{\partial p}{\partial y}, \tag{3}$$

and

$$\left(\frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x}\right) w = - \frac{1}{\rho_o} \frac{\partial p}{\partial z} - \frac{g\rho}{\rho_o}, \tag{4}$$

the continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{5}$$

and the incompressibility condition

$$\left(\frac{\partial}{\partial t} + u_o \frac{\partial}{\partial x}\right) \rho - \frac{N^2}{g} \rho_o w = 0. \tag{6}$$

If, for simplicity, we set f and $u_o$ to be constants, then equations (2)-(6) admit plane wave solutions of the form

$$\left[\begin{Bmatrix} u \\ v \\ w \\ p \\ \rho \end{Bmatrix}\right] = Re \left[\begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ p_1 \\ \rho_1 \end{Bmatrix} : \exp\left\{i(kx + ly + mz - \omega t)\right\}\right] \tag{7}$$

where *Re* denotes real part, where $\omega$ satisfies

$$(\omega - ku_o)^2 = \frac{N^2 \ (k^2 + l^2) + f^2 \ m^2}{(k^2 + l^2 + m^2)}, \tag{8}$$

the well-known dispersion relation for inertia-gravity waves.

## KELVIN WAVES IN UNIFORM FLOW

Other physically-meaningful solutions of equations (2)-(6) can be found for certain systems, of particular interest being those for which

$$v = 0 \tag{9}$$

everywhere. Writing

$$\begin{Bmatrix} u \\ w \\ p \\ \rho \end{Bmatrix} = Re \left[\begin{Bmatrix} u_1 \ (y) \\ w_1 \ (y) \\ p_1 \ (y) \\ \rho_1 \ (y) \end{Bmatrix} \cdot \exp\left\{i(kx + mz - \omega t)\right\}\right] \tag{10}$$

and eliminating $u_1$ from equations (2) and (3), we find for $p_1$ the differential equation

$$\frac{\partial p_1}{\partial y} + \frac{kf}{(\omega - ku_o)} \cdot p_1 = 0, \tag{11}$$

which integrates to

$$p_1 = P.\ \exp \left\{ \frac{-k}{(\omega - ku_o)} \int^y f(y^1).\ dy^1 \right\} \tag{12}$$

where $y^1$ is a dummy variable (supposing now that f is not necessarily independent of y). Then consideration of equations (4) to (6) shows that the other dependent variables $u_1$, $v_1$ $\rho_1$ exhibit a similar y –dependence and that the "Kelvin-wave" defined by equations (9) and (10) is a solution if $\omega$ satisfies

$$(\omega - ku_o)^2 = \frac{N^2 k^2}{(k^2 + m^2)}. \tag{13}$$

This id identical to the dispersion relationship for two-dimensional internal gravity waves in a non-rotating fluid. Kelvin waves evidently differ from inertia-gravity waves in the role played by Coriolis forces due to rotation. In the latter case Coriolis forces increase the wave frequency and render the trajectories of individual fluid elements elliptical rather than linear, whilst in the former case Coriolis forces modify the pressure field so as to constrain individual fluid elements everywhere to move in straight lines in planes perpendicular to the y-axis, at a speed which varies exponentially with y at a rate proportional to f (see equation (12)). Within the framework of the formal theory of inertia-gravity waves, Kelvin waves correspond to solutions having a wavenumber l in the y direction imaginary and satisfying

$$\ell = \frac{ikf}{(\omega - ku_o)} \tag{14}$$

(see equations (8) and (13)), where we must choose that sign of $(\omega - ku_o)/k$ in equation (13) which excludes unbounded solutions, namely that for which

$$\frac{kfy}{(\omega - ku_o)} > 0\ . \tag{15}$$

*Edge waves.* Consider a fluid bounded on $y = 0$ by vertical rigid impermeable wall and unbounded as $y \to \infty$. The boundary condition $v = 0$ is automatically satisfied by (9) and if f has the same sign everywhere then from equation (12) or (15), boundedness requires that

$$\mathrm{sgn}(-u_o + \omega/k) = \mathrm{sgn}\, f \tag{16}$$

In more general terms, if $\mathbf{n}$ is a vector normal to the boundary wall and directed towards the fluid and if $\mathbf{f} = (o, o, f)$, then relative to the mean flow the horizontal phase velocity is in the direction $\mathbf{n} < \mathbf{f}$.
*Equatorial Kelvin waves.* Kelvin waves may exist even in an unbounded system if f does not have the same sign everywhere. If the y-coordinate is defined such that $f > o$ as $y \to \infty$ and $f < o$ in $y < o$, with $f = o$ on the "equator" $y = o$, then boundedness requires that $\omega/k > u_o$, corresponding to eastward phase propagation relatively to the basic flow. An important recent development in dynamical meteorology has been the observational and theoretical study of eastward-propagating equatorial Kelvin waves in the atmosphere (Holton and Lindzen, 1968, Wallace and Kousky, 1968, Wallace, 1973).

## KELVIN WAVES IN A SHEAR FLOW

The dispersion relations (8) and (13) may be rewritten

$$m^2 = \frac{\{N^2 - (\omega - ku_o)^2\}(k^2 + \ell^2)}{(\omega - ku_o)^2 - f^2} \tag{17}$$

for inertia-gravity waves, and

$$m^2 = \frac{\{N^2 - (\omega - ku_o)^2\}\; k^2}{(\omega - ku_o)^2} \tag{18}$$

for the Kelvin waves. Rough arguments (note that (17) and (18) are for constant $u_o$) suggest that a critical level is located where $m^2 \to \infty$, i.e. where

$$(\omega - ku_o)^2 = f^2 \tag{19}$$

for the inertia-gravity waves, but where

$$(\omega - ku_0)^2 = 0 \tag{20}$$

for the Kelvin waves, just as for non-rotating internal gravity waves. Equations (19) and (20) are confirmed for rotating and non-rotating gravity waves by Jones (1967) and by Booker and Bretherton (1967), respectively. While we shall not calculate here the properties of the waves across the critical level, the WKB analysis presented below does demonstrate that the usual singularity exists in the solution for a Kelvin wave in a shear flow where equation (20) is satisfied.

The propagation of Kelvin waves through a shear flow on an equatorial beta-plane has been investigated by Lindzen (1971), though there was, in this case, no distinction between the two cases (19) and (20). Nevertheless, the analysis presented below is very similar in its details to that of Lindzen, and so the presentation of the mathematics will be kept to a minimum.

Consider a system rotating with constant angular velocity f/2, bounded by a rigid wall on $y = o$ and unbounded as $y \to \infty$. The basic state of the fluid is that of a shereared mean flow $u_0$ (z). Writing

$$\tilde{\omega}(z) = \omega - ku_0(z) \tag{21}$$

we find that equations (2) – (6) admit solutions of the form

$$p = \rho_0 \cdot Re \ \{P(y,z) \exp [i(kx - \omega t)]\} \tag{22}$$

if

$$\frac{\partial^2 P}{\partial z^2} - \frac{(\tilde{\omega}^2 + N^2)}{(\tilde{\omega}^2 - N^2)} \left(\frac{1}{\tilde{\omega}} \frac{d\tilde{\omega}}{dz}\right) \frac{\partial P}{\partial z} - \frac{(\tilde{\omega}^2 - N^2)}{(\tilde{\omega}^2 - f^2)} \cdot (k^2 - \frac{\partial^2}{\partial y^2}) P = 0. \tag{23}$$

With

$$V = Re \left\{ V(y,z) \exp \left\{ i(kx - \omega t) \right\} \right\} \tag{24}$$

equations (2) and (3) give

$$V = \frac{i}{(\tilde{\omega}^2 - f^2)} (kf - \tilde{\omega} \frac{\partial}{\partial y}) P \cdot \tag{25}$$

Equation (23) is, in general, intractable, but if the coefficient of P in the third term on the left and side of the equation is large compared with $|u_o^{-1} \partial^2 \tilde{u}_o / \partial z^2|$ the vertical scale of P is much smaller than that of $u_o$ and the problem may be solved using the WKB method. The simplest case for which this procedure is appropriate is that of a strongly stratified fluid, such that

$$\frac{f}{N} = \epsilon \tag{26}$$

and

$$\frac{\tilde{\omega}}{N} = \epsilon \quad \sigma(z) \tag{27}$$

where $\epsilon \ll 1$ and $\sigma(z)$ is of order unity. Then equation (23) may be written

$$\frac{\partial^2 P}{\partial z^2} + \frac{(1 + \epsilon^2 \sigma^2)}{(1 - \epsilon^2 \sigma^2)} \cdot \left( \frac{1}{\sigma} \frac{d\sigma}{dz} \right) \frac{\partial P}{\partial z} + \frac{1}{\epsilon^2} \frac{(1 - \epsilon^2 \sigma^2)}{(\sigma^2 - 1)} (k^2 -$$

$$\frac{\partial^2}{\partial y^2} ) P = 0. \tag{28}$$

This equation may now be readily solved by expanding P(y,z) in the form

$$P = P_o \exp\left\{\frac{1}{\epsilon}\left[F_o(y,z) + \epsilon F_1(y,z) + 0(\epsilon^2)\right]\right\} \tag{29}$$

where $P_o$ is a constant. The boundary condition on $y = 0$ may be expressed, from equations (25) and (29), as

$$\left.\begin{aligned} &\frac{\partial F_1}{\partial y} - \frac{k}{\sigma} = 0 \\ &\text{and} \\ &\frac{\partial F_n}{\partial y} = 0\ ,\ n \neq 1. \end{aligned}\right\} \tag{30}$$

Substituting expression (29) into (28) and equating coefficients of the exponential term at each order in $\epsilon$, it is straightforward (see Lindzen, 1971) to find the solution, subject to boundary conditions (30) and that of boundedness as $y \to \infty$, to be

$$P = P_o\ [\sigma(z)]^{1/2} \exp\left\{\pm \frac{ik}{\epsilon}\int^z \frac{dz^1}{\sigma(z^1)} - \frac{ky}{\sigma(z)} + 0\ (\epsilon)\right\}, \tag{31}$$

the $\pm$ signs representing waves with upward and downward phase speeds, respectively, if $\sigma > 0$.

## DISCUSSION

It is clear from equation (31) that the singularity in the solution occurs where $\sigma = 0$, i.e. where equation (20) is satisfied. As the wave approaches this level, the vertical "wavelength" which, in (31), we identify with $\epsilon\sigma/k$, tends to zero. Note that there is no singularity where $\tilde{\omega} = \pm f$, i.e. $\sigma^2 = 1$, despite the singularity in equation (28).

In these respects, the behaviour of the wave is identical to that of an internal gravity wave in the region of its critical level. However, note the meridional structure of the wave, which is of the form $\exp\{- kfy/\ (\omega - ku_o)$ ; the meridional extent of the disturbance decreases (tending to zero– as the critical level is approached. Indeed, across that level, $\sigma$ changes sign, and the wave cannot exist there (being unbounded as $y \to \infty$). Hence it seems unlikely (in the absence of non-linear effects) that there can be any leakage across the critical level, unlike the case of internal gravity waves (though the leakage is very small if the Richardson number is moderately large; Booker and Bretherton, 1967). The exact structure in the region of the critical level cannot be determined by the present limited analysis.

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## *COMUNICATION*

## *MODERN STANDARDS FOR GRAVITY SURVEYS*

The modern gravity meter is capable of determining difference in gravity between measurement sites with very high accuracy. However, the gravity values established by one survey can be related to those established by another survey only when both are tied to gravity base stations whose gravity values are in a uniform system.

The benefits of adjusting all measured gravity values to a single datum are manifold. The goedesist, for example, must have world-wide coverage of gravity measurements related to a uniform system for computation of the earth's level surface configurations. The geologist and geophysicist benefit by being able to use data from several different surveys for uniform structural interpretations on a local, regional and global scale. Having local gravity surveys which are compatible with compilations over broader areas without systematic errors due to datum shifts is essential for regional-residual separations and interpretations over sedimentary basins and crustal blocks. With today's emphasis on economy in all operations, the ability to trade, buy, and/or use existing data in conjunction with new surveys can lead to important cost savings.

The absolute gravity datum adopted and recommended for all gravity surveys by the International Union of Geodesy and Geophysics is defined by the gravity values at more than 1800 measurement sites which comprise the International Gravity Standardization 1971 (IGSN71).[1]

In addition to providing an absolute datum for gravity measurements, stations of the IGSN71 are strongly recommended for use in gravimeter calibration.

The readjustment of national gravity networks to IGSN71 datum and scale has been completed in many but not all countries. Therefore, users are cautioned to verify that the base reference values used for any new gravity survey are referred to IGSN71. Verification can be accomplished by contacting the national agency responsible for gravity standards. In case of doubt, new surveys should be tied directly to IGSN71 stations. Gravity values and site descriptions for all IGSN71 stations, as well as and index map showing their location, are available on request from the International Gravity Bureau (IGB).[2]

It is very important to recognize that, for calculation of gravity anomalies, the use of IGSN71 for observed gravity must be accompanied by the use of the Geodetic Reference System 1967[3] for computation of theoretical gravity.

C. Morelli, Presidente
International Gravity Commission

[1] Morelli C., Gantar C., Honkasalo T., McConnell R. K., Tanner I. G., Szabo B., Uotila U., Whalen C. T., 1971: *The International Gravity Standardization Net 1971.* Int. Ass. of Geodesy, 39ter Rue Gay Lussac, 75005 Paris, $ 10, 194 pp.

[2] Bureau Gravimétrique International, 9 Quai St. Bernard, Tour 14, Paris (5e).

[3] Int. Ass. of Geodesy, 1971: *Geodetic Reference System 1967.* Publ. Spec. Bull. Géod., 116 pp.