TRANSIENT ELECTROMAGNETIC RESPONSE OF A CONDUCTING INFINITE CYLINDER EMBEDDED IN A CONDUCTING MEDIUM

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RESUMEN

Se obtiene en la forma de integrales definidas la respuesta electromagnética transitoria de un cilindro conductor permeable, incrustado en un espacio conductor infinito. La fuente es un cable infinito aislado que yace fuera del cilindro y que lleva una corriente de Heaviside. Las corrientes de desplazamiento no han sido tomadas en consideración. También se proporciona una expresión de la función de respuesta transitoria bajo la aproximación quasi-estática.

ABSTRACT

Transient electromagnetic response of a conducting permeable cylinder embedded in a conducting infinite space is obtained in the form of definite integrals. The source is an infinite insulated cable which lies outside the cylinder and carries a Heaviside current. The displacement currents have been neglected. Expression for transient response function under quasi-static approximation is also given.

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INTRODUCTION

Recently it has been pointed out by several authors (e. g. Ward, 1971; Singh, 1972) that the effect of finite conductivity of the host rock must be considered in electromagnetic (e.m.) exploration of conductive massive sulphide ore bodies. This is particularly true for time-domain e.m. methods where, theoretically, all the freguencies are present. The author (Singh, 1972) has considered the transient response of a conducting, permeable sphere embedded in a conducting infinite space under an arbitrarily oriented magnetic dipole exitation. In this paper the transient response of a conducting, permeable and homogeneous infinite cylinder embedded in a conducting infinite space is considered. The source is an infinite insulated cable which lies parallel to the axis of the cylinder and carries a Heaviside current. The contributions from displacement-current have been neglected. The transient response, under quasi-static approximation is also considered.

We shall obtain the time-domain solution by taking the inverse transform of the known frequency-domain solution which is given by Wait (1952).

Frequency-domain solutions have also been given by Kertz (1960) for a homogeneous cylinder under a uniform field, and by Negi (1962) and Negi et al. (1972) for inhomogeneous cylinders under uniform field and line source, respectively. These results have been specialized to the quasi-static case. Verma (1972) has given quasi-static time-domain response of a homogeneous cylinder under uniform field of step and ramp type excitation.

An infinitely long current carrying cable in a conductive medium is a difficult source to realize in practice but the result of this paper may be useful (a) in evaluation of the validity of quasi-static approximation, (b) in construction of solution of a more realistic situation in which the source is on top of a conducting half space containing a cylindrical ore body, and (c) in marine exploration.

FORMULATION AND SOLUTION

Let an isotropic. homogeneous, infinite cylinder of radius a and of electrical properties σ_2 μ_2 and ϵ_2 be embedded in an infinite space of electrical propertier σ_1 μ_1 and ϵ_1. An infinitely long insulated cable, carrying current, lies at point $S(r_o,\phi_o)$ parallel to the axis of the cylinder (Fig. 1). The point S may be outside the cylinder ($r_0 > a$) or inside the cylinder ($r_0 < a$), Here we shall consider the former case only giving the secondary magnetic components in the outer medium The latter case can be treated in exactly a similar fashion. The expressions for the primary field components are given in Appendix A.

A. FRECUENCY-DOMAIN

Assuming a harmonically time-varying current $I\, e^{i\omega t}$ in the cable, the secondary magnetic field components in the outer medium, in MKS system of units, can be written as (see Wait, 1952 for details):

$$H_r^s(\omega) = \frac{I\, e^{i\omega t}}{2\pi r} \sum_{n=1}^{\infty} \frac{\sin n(\phi - \phi_o)}{(bd)^n} \left[h_{nr}^s(z) \right] \tag{1}$$

$$H_\phi^s(\omega) = -\frac{I\, e^{i\omega t}}{2\pi r} \sum_{n=0}^{\infty} \frac{\cos n(\phi - \phi_o)}{(bd)^n} \left[h_{n\phi}^s(z) \right] \tag{2}$$

where,

$$h_{nr}^s(z) = 2n\,(bd)^n\, K_n(Cdz)\, \frac{N_n(z)}{D_n(z)}\, K_n(Cbz) \tag{3}$$

$$h_{n\phi}^s(z) = -\delta_n\,(bd)^n\, Cbz\, K_n(Cdz)\, \frac{N_n(z)}{D_n(z)}\, K_n'(Cbz) \tag{4}$$

$$\frac{N_n(z)}{D_n(z)} = \left[\frac{I_n(Cz)\, I_n'(z) - CK\, I_n'(Cz)\, I_n(z)}{K_n(Cz)\, I_n'(z) - CK\, K_n'(Cz)\, I_n(z)} \right] \tag{5}$$

$$C^2 = \frac{\gamma_1^2}{\gamma_2^2} = \left[\frac{i\sigma_1 \mu_1 \omega - \epsilon_1 \mu_1 \omega^2}{i\sigma_2 \mu_2 \omega - \epsilon_2 \mu_2 \omega^2} \right] \tag{6}$$

$$z^2 = \gamma_2^2 \, a^2 = (i\sigma_2 \mu_2 \omega - \epsilon_2 \mu_2 \omega^2) \, a^2 \tag{7}$$

$$K = \mu_2 / \mu_1 \tag{8}$$

$$b = \frac{r}{a} \tag{9}$$

$$d = \frac{r_o}{a} \tag{10}$$

$$\delta_n = \begin{cases} 1 & \text{for } n = 0 \\ 2 & \text{for } n \geqslant 1 \end{cases} \tag{11}$$

I_n (z) and K_n (z) are modified Bessel functions.

If the conduction currents are much greater than the displacement currents, i.e. $\epsilon_j \; \omega \ll \sigma_j$ (j = 1, 2), as is very often the case in exploration geophysics, them the expressions for C and z given in equations (6) and (7) simplify to:

$$C^2 = \left(\frac{\sigma_1 \mu_1}{\sigma_2 \mu_2} \right) \tag{12}$$

and,

$$z^2 = i\omega\beta^2 \tag{13}$$

where,

$$\beta^2 = \sigma_2 \mu_2 a^2 \tag{14}$$

Applying asymptotic expansions of modified Bessel functions, it can be easly shown that, under quasi-static approximation ($|Cz|$, $|Cbz|$, $|Cdz| \ll 1$),

$$h^s_{nr}(z),\ h^s_{n\phi}(z) \longrightarrow R_n(z)\ ;\ n \geqslant 1$$

where,

$$R_n(z) = \left[\frac{z\, I'_n(z) - nK\, I_n(z)}{z\, I'_n(z) + nK\, I_n(z)} \right] \tag{15}$$

which is the quasi-static response function. Note that there is an error in the quasi-static response function given by Wait (1952).

B. TIME-DOMAIN

Given secondary magnetic field components, under harmonic excitation, the transient response can be written as:

$$H^s_r(t) = \frac{I}{2\pi r} \sum_{n=1}^{\infty} \frac{\sin n(\phi - \phi_o)}{(bd)^n} \left[h^s_{nr}(t) \right] \tag{16}$$

$$H^s_\varphi(t) = \frac{-I}{2\pi r} \sum_{n=0}^{\infty} \frac{\sin n(\phi - \phi_o)}{(bd)^n} \quad [h^s_{n\phi}(t)] \tag{17}$$

with,

$$h^s_{nr}(t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} h^s_{nr}(s)\, \phi(s)\, e^{st}\, ds \tag{18}$$

$$h^s_{n\phi}(t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} h^s_{n\phi}(s)\, \phi(s)\, e^{st}\, ds \tag{19}$$

where $s = i\omega$, ϵ is the real positive constant and $\phi(s)$ is the Laplace transform of the primary input pulse. In this paper we shall assume a Heaviside pulse ($\phi(s) = \frac{1}{s}$).

We shall neglect the contribution from displacement currents. Thus the solution would be valid for $t \gg \epsilon_j/\sigma_j (j = 1, 2)$. From equation (13) we note that

$$z = s^{1/2}\, \beta \tag{20}$$

It is easly shown that the integrands of equations (18), and (19), are double-valued functions of complex s. In order to make these integrands single-valued functions of s, so that we can evaluate the integrals using residue theorem, we introduce a branch cut along negative real s axis as shown in Fig. 2 and require that $-\pi < \arg s \leqslant \pi$. Now, considering the contour of Fig. 2, we have

$$\oint = 2\pi i[\text{sum of the residues}] \tag{21}$$

It is not difficult to show that there are no poles of the integrands in or on the contour (see Singh, 1972 for a proof of a similar problem) and that the integral over the large circular arcs BC and FA vanish in the limit as the radius R → ∞ Thus

$$\lim_{R\to\infty}\left[\int_A^B = \int_D^C + \int_F^E + \int_E^D\right]$$

The integral around ED as $\delta \to O$ gives the static part of the solu· tion from the pole at the origin (which is also a branch point). But this part of the solution is not of interest in exploration and, therefore, we shall ignore it. Considering equation (18) in detail und substituting

on DC $s = \dfrac{u^2}{\beta^2}\, e^{i\pi}$,therefore, $z = u\, e^{i\pi/2}$

and on FE $s = \dfrac{u^2}{\beta^2}\, e^{-i\pi}$,therefore, $z = u\, e^{-i\pi/2}$

we can write

$$\lim_{R\to\infty}\left[\frac{1}{2\pi i}\int_D^C h^s_{nr}(s)\,\frac{e^{st}}{s}\,ds\right]$$

$$= \frac{1}{2\pi i}\int_O^\infty 4n\,(bd)^n\, K_n\,(iCdu)\,\frac{N_n\,(iu)}{D_n\,(iu)}\,K_n\,(iCbu)\; e^{-u^2t/\beta^2}\,\frac{du}{u} \tag{22}$$

Using the following properties of modified Bessel functions

$$K_n \left[z\, e^{\pm i \frac{\pi}{2}} \right] = \frac{\pm \pi}{2}\, i e^{\mp i \frac{\pi n}{2}} \left[- J_n (z) \pm i\, Y_n (z) \right] \qquad (23)$$

$$I_n \left[z\, e^{\pm \frac{i\pi}{2}} \right] = e^{\pm \frac{i \pi n}{2}} \left[J_n (z) \right] \qquad (24)$$

we can write equation (22) in the following form:

$$- n\,(bd)^n \int_0^{\infty} H_n^{(2)} (Cdu)\, \frac{P_n(u)}{Q_n(u)}\, H_n^{(2)} (Cbu)\, e^{-u^2 t/\beta^2}\, \frac{du}{u} \qquad (25)$$

where,

$$P_n (u) = J_n (Cu)\, J_n^{'} (u) - CK\, J_n^{'} (Cu)\, J_n (u) \qquad (26)$$

$$Q_n (u) = H_n^{(2)} (Cu)\, J_n^{'} (u) - CK\, H_n^{(2)'} (Cu)\, J_n (u) \qquad (27)$$

Similarily we can show that

$$\lim_{R \to \infty} \left[\frac{1}{2\pi i} \int_F^E h_{nr}^s (s)\, e^{st}\, \frac{ds}{s} \right]$$

$$= - n\,(bd)^n \int_0^{\infty} H_n^{(1)} (Cdu)\, \frac{P_n (u)}{Q_n (u)}\, H_n^{(1)} (Cbu)\, e^{-u^2 t/\beta^2}\, \frac{du}{u} \qquad (28)$$

where bar on Q_n (u) represents complex conjugate of Q_n (u).

Now, h^s_{nr} (t), ignoring the static part, can be obtained by summing equations (25) and (28) and can be written as:

$$h^s_{nr}(t) = -2n(bd)^n \,\mathrm{Re}$$

$$\left[\int_0^\infty H_n^{(2)}(Cdu) \frac{P_n(u)}{Q_n(u)} H_n^{(2)}(Cbu)\, e^{-u^2 t/\beta^2} \frac{du}{u} \right] \quad (29)$$

Following a similar procedure as given above, we can write:

$$h^s_{n\phi}(t) = \delta_n (bd)^n \,\mathrm{Re}$$

$$\left[\int_0^\infty Cb\, H_n^{(2)}(Cdu) \frac{P_n(u)}{Q_n(u)} H_n^{(2)\prime}(Cbu)\, e^{-u^2 t/\beta^2}\, du \right] \quad (30)$$

where, Re in equations (29) and (30) means that the real part of the expression in the brackets must be taken.

Equations (29) and (30) give the transient response functions for a Heaviside input pulse when displacement-currents are negligible. These are a function of following dimensionless parameters:

$$t/\beta^2,\ K,\ C,\ Cb,\ Cd$$

In numerical integration one must remember that the zeros of Q_n (u) lie very close to the line of integration for small values of C. Since most of the contribution to the integrals come from this region, special care must be taken.

QUASI-STATIC TRANSIENT RESPONSE FUNCTION

Quasi-static transient response function, $R_n(t)$, for a Heaviside input pulse is given, from equation (15), by:

$$R_n(t) = \frac{1}{2\pi i} \int_{\epsilon - i\infty}^{\epsilon + i\infty} R_n(s)\, e^{st} \frac{ds}{s} \qquad (31)$$

where $R_n(s)$ is obtained from equation (15) by noting the relation given in equation (20). We shall consider non-permeable (K = 1) and permeable (K ≠ 1) cylinder cases separately:

Non-Permeable Case (K = 1):

It is easily shown that $R_n(s)$ is a single-valued function of s and has simple poles at s = 0 (where the residue is zero) and at the zeros of

$$I_{n-1}(z) = 0 \qquad (32)$$

which are all imaginary. If $z = iy_{n-1,\,j}$ is a zero, then $z = -iy_{n-1,\,j}$ is also a zero. By applying residue theorem and recurrence relations of modified Bessel functions and noting the relation given in equation (24) we can show that

$$R_n(t) = 4n \sum_{j=1}^{\infty} \frac{e^{-y^2_{n-1,j} t/\beta^2}}{y^2_{n-1,j}} \qquad (33)$$

where $iy_{n-1,\,j}$ is a zero of $I_{n-1}(z)$. When n = 1, the result is same as given by Verma (1972).

Permeable Case (K ≠ 1):

R_n (s) for this case also is a single valued function of complex s and has simple poles at s = 0 (which gives static part of the solution) and at the zeros of

$$\Delta_n(z) = z\, I_n'(z) + n\, K\, I_n(z) = 0 \tag{34}$$

which are all imaginary, if $z = iy_{n,j}$ is a zero, then $z = -iy_{n,j}$ is also a zero. Using residue theorem and the properties of modified Bessel function and neglecting the static part of the solution, we can show that

$$R_n(t) = 4n^2\, K\,(K-1) \sum_{j=1}^{\infty} \frac{J_n(y_{n,j})\, e^{-y_{n,j}^2\, t/\beta^2}}{y_{n,j}\,\{n^2(1-K^2) - y_{n,j}^2\}\; J_{n-1}(y_{n,j})} \tag{35}$$

where, $iy_{n,j}$ is a zero of $\Delta_n(z)$.

Computation of R_n (t) is straight forward. Zeros of I_n (z) can be looked in any book on Beseel functions and the zeros of Δ_n (z) can be easily determined with the help of a computer.

CONCLUSIONS

Whereas only one response function, $R_n(t)$, is needed under quasi-static approximation, two response functions, $h^s_{nr}(t)$ and $h^s_{n\phi}(t)$, are needed to describe the transient secondary magnetic field components when the finite outer conductivity is considered. $R_n(t)$, in general, is a function of two dimension-less parameters: t/β^2 and K; whereas $h^s_{nr}(t)$ and $h^s_{n\phi}(t)$ are, if displacement-currents are negligible, a function of five dimension-less parameters: t/β^2, K, C, Cb and Cd. Also the computation of $h^s_{nr}(t)$ and $h^s_{n\phi}(t)$ is more difficult than $R_n(t)$. Clearly, the introduction of finite outer conductivity increases mathematical and interpretational difficulties very considerably. However, it may be unavoidable in the time-domain and, therefore, the exact case must be considered first and compared with the quasi-static aproximation before using the former for interpretation.

Numerical results will be presented in another paper.

APPENDIX A

For an infinite line source located at point S (Fig. 1) inside a conductive medium and carrying a harmonically time-varying current $Ie^{i\omega t}$ the field components are given by (Wait, 1952):

$$E_{z'}(\omega) = -\frac{i\omega\mu_1 \, Ie^{i\omega t}}{2\pi} K_o \, [i\sigma_1 \mu_1 \omega)^{1/2} \, r'] \qquad (A.1)$$

$$H_{\phi'}(\omega) = \frac{(i\sigma_1 \mu_1 \omega)^{1/2} \, Ie^{i\omega t}}{2\pi} K_1 \, [(i\sigma_1 \mu_1 \omega)^{1/2} \, r'] \qquad (A.2)$$

if the displacement currents are neglected.

If the line source is excited by a Heaviside pulse, the transient field componentes, obtained by taking Laplace transform of (A.1) and (A.2) appropriately, are given by:

$$E_{z'}(t) = -\frac{\mu_1}{4\pi} \, I \, \frac{e^{-\frac{\sigma_1 \mu_1 r'^2}{4t}}}{t} \qquad (A.3)$$

$$H_{\phi'}(t) = \frac{I}{2\pi r'} \, e^{-\frac{\sigma_1 \mu_1 r'^2}{4t}} \qquad (A.4)$$

Fig I

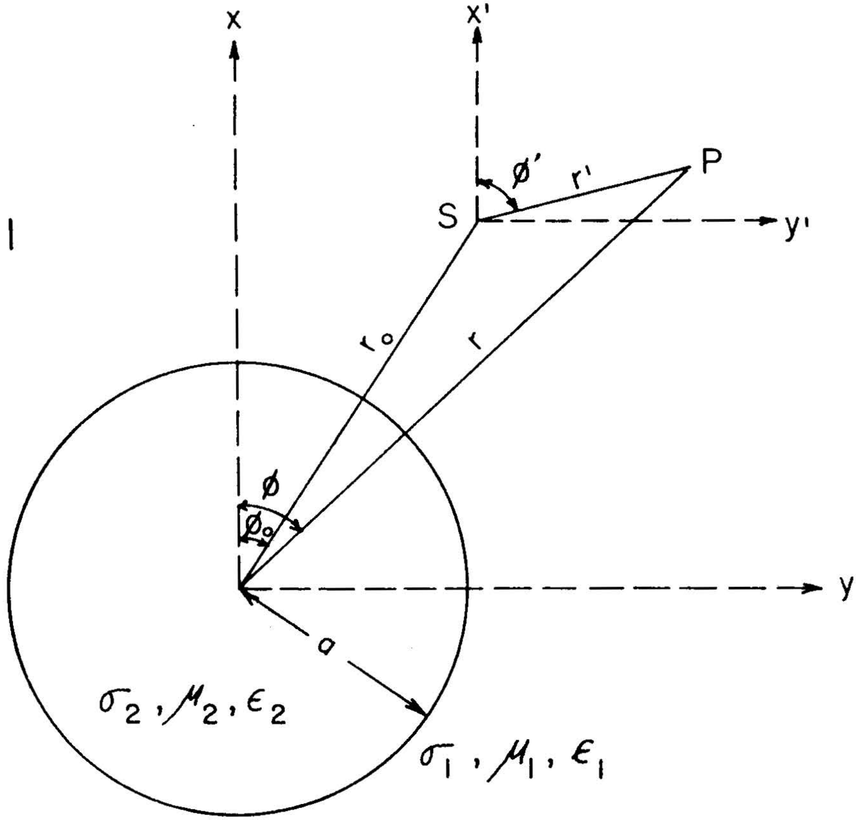

Figure 1: A permeable conducting infinite cylinder in a conducting infinite space. The line source is located at S (r_o, ϕ_o) outside the cylinder.

Fig 2

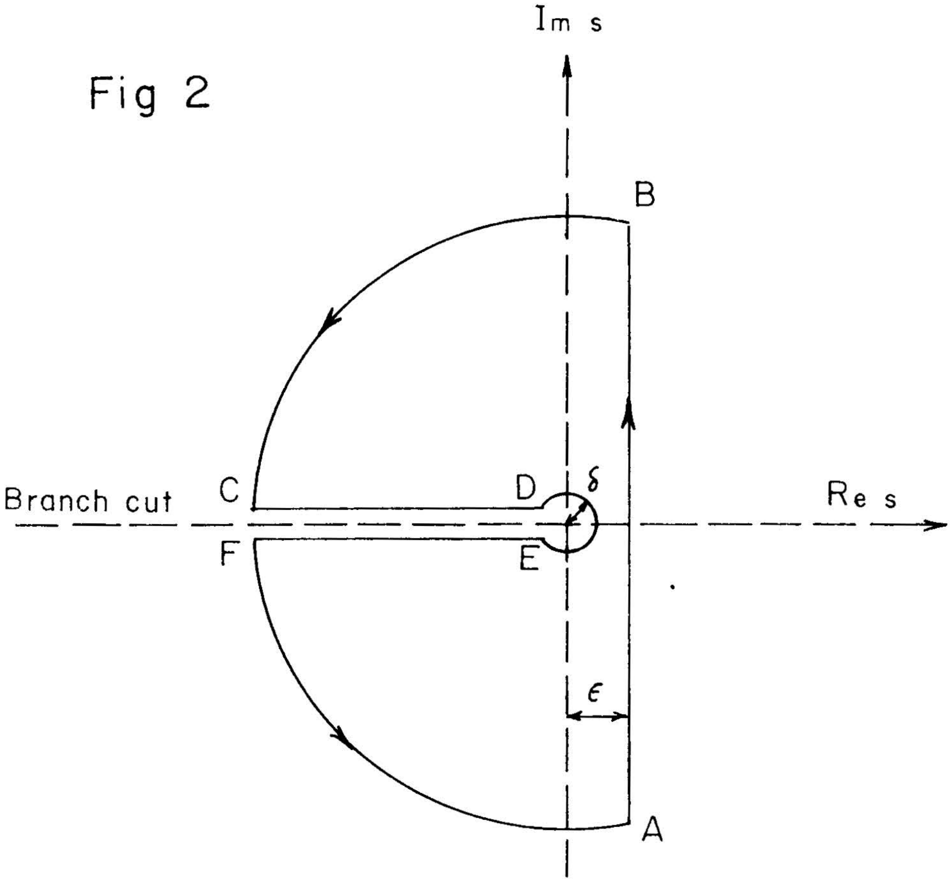

Figure 2: Integration contour in complex s plane. Branch cut is along negative real s axis and $-\pi < \arg s \leqslant \pi$. No poles lie inside and upon the closed contour.

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