GEOFISICA INTERNACIONAL

REVISTA DE LA UNION GEOFISICA MEXICANA, AUSPICIADA POR EL INSTITUTO DE GEOFISICA DE LA UNIVERSIDAD NACIONAL AUTONOMA DE MEXICO

Vol. 15 | México, D. F., 1o. de julio de 1975 | Núm. 3

FORMULATION OF THE THEORY OF PERTURBATIONS FOR COMPLICATED MODELS. PART II: WEATHER PREDICTION

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RESUMEN

En este artículo se presenta la aplicación de funciones conjugadas y teoría de perturbaciones al problema de predicción del tiempo.

ABSTRACT

This paper deals with the application of conjugate functions and perturbation theory to the problem of weather prediction.

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FORMULATION OF THE THEORY OF PERTURBATIONS FOR COMPLICATED MODELS

Weather prediction is one of the central scientific problems that deserves special attention because of its great importance to many branches of national economy and first of all to agriculture, aircraft, communication and so on. The improvement of the existing weather forecasting capabilities immediately tells upon the efficiency of national economy. Economical reserves gained as a result of a moderate improvement in the quality of weather prediction will effect a saving of hundreds of millions roubles annually. This means that intensive research in the field of weather prediction will make possible a creation of conditions for big economical effect throughout the country.

At the same time this is one of the most complicated and interesting scientific problems because it involves basic concepts of hydrodynamics, laws of statistical mechanics and physics, methods of radiative transfer theory, numerical algorithms.

Weather forecast for a period of over one week ahead is usually called extended-range prediction. It, too, has its own predictability limits, i.e. two weeks, a month, a season. The basic factors and the principal computational scheme depend on the prediction range. However there are some characteristics that such forecasts have in common, i.e. they are global and depend much on the thermal state of the continents and the World Ocean.

The problem of the extended-range weather prediction is very important, that is why it will be of supreme concern to researchers during the forthcoming decades. This problem can be justifiably called a "problem of our age" because of the lack of the necessary meteorological and hydrophysical data and because it requires extremely complicated physical formulation and methods of solution. This implies considerable attention to further development of some sectors in the physics of the atmosphere and ocean, satellite meteorology and powerful computers. There are at least three extended-

-range weather prediction methods. These are synoptic, statistical and hydrodynamic methods.

The synoptic method is based on qualitative typification of similar atmospheric processes in their initial state and prehistoy. It uses a certain set of laws which have been observed by meteorologists in their everyday work and which reflect the basic natural developments.

In the practice of the extended-range weather prediction the routine synoptic methods are currently in use.

The statistical method began to be used in long-range weather prediction not so long ago, in the early 1960's. At first it was based on representation of the fields of meteorological elements in terms of Fourior series expansion in spherical harmonics and on time extrapolation of the Fourier coefficients which were known at previous instants of time. Later statistical methods began to use natural orthogonal functions the result of the solution of the spectral problem for correlation matrix. The system of natural orthogonal functions generated by this spectral problem appeared to contain much useful information and make possible a fairly good representation of meteorological fields by a small number of terms. This approach to representation of fields with subsequent time extrapolation of coefficients is the basic concept of the statistical method.

The central point of the present report are hydrodynamic methods of extended-range weather prediction and application of the perturbation theory which we are going to discuss now.

Of great importance for the understanding of the extended-range prediction methods were the studies carried out by the American scientists Smagorinsky, Leith, Mintz and Lorentz, dealing with simulation of the general circulation of the atmosphere. Though these studies were not aimed at weather prediction, from them derivated some of the most important conclusions about meteorological elements predictability, the atmosphere-ocean fluctuations, the time of stablishment of periodic atmospheric and oceanic processes and so on. However an attempt to use their equations for prediction purposes from the initial actual fields was fairly pessimistic, as it appeared that the predictability term of meteorological processes by these schemes was about two weeks only. After repeated verifications of this fact it was concluded that weather prediction for a period of over two weeks ahead was not possible at all.

Investigations of Soviet meteorologists show that it is possible to give more general statements of the hydrodynamic weather prediction problem with whose aid one can extend the prediction term up to a season. Mathematical apparatus of this approach rests on the use of conjugate hydrodynamic equations and perturbation theory specially constructed. This approach has been developed at the Computer Center of the Siberian Branch of the Soviet Academy of Sciences.

The principal concept of formulation of the extended-range weather prediction problem is as follows.

The most important mechanism responsible for formation of weather anomalies against the seasonal atmospheric behaviour is planetary cloudiness which creates conditions for an irregular warming of the continents and the World Ocean.

The irregularly warmed regions of the continents and the ocean affect hydrodynamical progress of atmospherical processes and cause more or less essential deviations of weather from climate. This process becomes more complicated because the irregularly warmed waters of the World Ocean due to circulation are transported with the currents to new, and sometimes fairly remote, geographical regions of the planet and give their heat back to the atmosphere causing, on the first glance, unexpected changes in weather.

In this complex process of conversion of the solar radiation to the kinetic energy of atmospheric motions there are observed direct and indirect relationships which eventually are responsible for long-period variations of weather conditions. These are nonlinear relationships and this is an important factor in maintaining the relative stability of atmospheric processes represented in a form of fluctuations.

Such nonlinear fluctuations were observed by investigators during the analysis of the actual observational data and the numerical experiments with the general circulation models.

Finally such atmospheric motions with regard for cloudiness and heat transfer were observed by Monin and Gavrilin in the simple linear model. They determined theoretically the period of the system fluctuation: for typical conditions it appeared to be equal to several months.

Thus, relying on the synoptic analysis, numerical simulation of the general circulation of the atmosphere and ocean and the analysis of dispersion relationships in the linearized circulation models, we can regard the main mechanism, responsible for long-term deviations of weather from climate, to have been established. This mechanism is founded on the interaction of the atmosphere with the ocean and the underlying surface of the continents with regard for the dynamics of the planetary cloud systems.

In general the formation of long-term weather anomalies can be presented as follows. Over the regions of the World Ocean there are cloud systems from wich the heat flux, going into the ocean surface layers, depends. If cloudiness is less than its climatic norm, then more solar radiation penetrates into the sea surface layer and it is warmed up more intensely.

Part of the heat stored by the ocean surface layer goes back to the atmosphere in a form of the long-wave radiation and warms it up. Another part is transported by means of the vertical turbulent exchange to lower layers of several hundreds meters deep. If the anomalous low cloudiness is set up for a long period of time (for a nonth or a season) a considerable warming of the ocean in this area is observed (of the order of 1° or several decimal points).

The warmed waters are transported by currents to the north of the Atlantic and the Pacific oceans. They reach the latitudes where in the ocean surface layer a zone of vertical instability is formed under the influence of the low air temperature. This causes generation of powerful convective irregular motions that release the storage of heat from the deep ocean layers to the atmosphere. The most intensive ocean-atmosphere heat exchange in the Atlantic occurs in the water area near Iceland, in the Pacific –near the Aleutian Islands. In the southern hemisphere it is the area of the Antarctic continent.

The heat of the ocean given up to the atmosphere warms up the air of the given region. Because of the neighbourhood of cold polar regions there arise big temperature contrasts which lead to formation of powerful cyclones that is a form of horizontal resolution of instability of the air masses —such cyclones are carried away to the east by the powerful mid-latitude planetary flow, transporting portions of heat to the continents and creating warm zones there. If we consider velocity of flows in the Atlantic and the Pacific and the removal of the active warm zone of the surface subtropic waters from northern critical areas of intensive heat exchange it appears that this process takes approximately as long as one season. It means that substantial temperature anomalies in the ocean will affect the weather of the continents in about a season.

It can be shown similarly that anomalous heavy cloudiness over the ocean eventually results in cooling of different continental regions.

In our scheme we abstracted from consideration of heat transfer in the atmosphere-continent system. Sometimes heavy heating of the continents by direct solar radiation with subsequent cooling also causes substantial temperature contrasts and transformation of atmospheric dynamical processes which results in appreciable weather anomalies.

However the effect of this factor continues, as a rule, no longer than two weeks. Therefore the problem of the atmosphere-continent interaction is closely related to weather prediction one month ahead.

The above representation is the consequence of mathematical analysis of basic and conjugate equations of the atmosphere and ocean dynamics. The analysis of these problems is presented below.

Usually it is very difficult to construct mathematical models simulating complicated processes and phenomena. Such models must incorporate many effects, not all of which are describable with required accuracy. This means that each time we use one or another simplified mathematical formulation of a problem, which allows us to describe only a few characteristic features of a process we omit many very important details. However such an approach toward mathematical simulation of physical processes is the basic instrument in our perception of natural phenomena which is being continuously refined.

In the present paper an attempt is made to present a more or less general approach towards construction of such mathematical models and their analysis.

To make our presentation a more explicit one, we will consider as an example the evaluation of the effect of different factors on climate and general circulation of the atmosphere, since this is the central problem in the study of the human environment. The climate of the Earth is known to us from observation. Though the climate of the planet changes, its changes are associated with long-period processes that manifest themselves distinctly after a period of many years. Therefore averaging of the data for many years that characterize climatic functionals of the atmosphere proves to be adequate for both description of the quasistationary climate and construction of the perturbation theory.

At the present time the preliminary evaluation of climatic modifications due to different factors, especially due to industrial human activities is sometimes a more important task than simulation of the climate proper. Therefore in the present paper we try to discuss different approaches to the construction of the perturbation theory with respect to the climate. We will start from the two fundamental statements. First, a system of equations of the atmosphere and ocean dynamics in its most complete form (details unknown) is capable to describe the climate of the planet's atmosphere. The climate is assumed to be known from observations. Second, perturbations of the climate are regarded to be small. The latter is equivalent to the condition of additivity of the climate and its perturbations. These two assumptions will alow us to formulate mathematical models for simulation of modifications of the climate.

On the whole the perturbation theory and evaluation of functionals of a problem are of a rather general character and as a rule they have very little to do with a specific problem. Therefore results of the present study are applicable to different mathematical models simulating complicated processes in physics, chemistry, biochemistry, engineering, etc.

Let us consider now a nonstationary problem in an abstract form

$$\frac{\partial \varphi}{\partial t} + A\varphi = f, \ \varphi = g \ \text{at} \quad t = 0 \tag{1}$$

which is put to correspond with

$$-\frac{\partial \varphi^*}{\partial t} + A^*\varphi^* = f^*, \ \varphi^* = g^* \quad \text{at} \ \ t = T. \tag{2}$$

Here f* and g* are as yet undefined vector functions to be chosen later. Equations (1), (2) are, respectively, multiplied by φ^*, φ, the results are subtracted one from the other and integrated over t on the interval $0 < t < T$. As a result we have

$$\int_o^T \frac{\partial}{\partial t} (\varphi^*, \varphi)dt + \int_o^T dt[(\varphi^*, A\varphi) - (\varphi, A^*\varphi^*)] =$$

$$= \int_o^T dt \ [(f, \varphi^*) - (f^*, \varphi)]. \tag{3}$$

Since A* and A are conjugate operators

$$(\varphi^*, A\varphi) - (\varphi, A^*\varphi^*) = 0,$$

expression (3) with consideration of initial conditions, reduces to

$$(g^*, \varphi_T) - (g, \varphi_o^*) = \int_o^T dt[f, \varphi^*) - (f^*, \varphi)],$$

where (4)

$$\varphi_T^* = \varphi^*\Big|_{t=T}, \ \varphi_o = \varphi \ \Big|_{t=o}$$

We now suppose that it is required to calculate the linear functional of solution (1) which can be presented as

$$J = (g^*, \varphi_T) + \int_o^T (f^*, \varphi)dt. \tag{5}$$

With the help of identity (3) this functional is written as

$$J = (g, \varphi_0^*) + \int_0^T (f, \varphi^*)dt. \quad (6)$$

Assume that the imput data of (1) are somewhat perturbed, i.e. in place of g and f, we consider g' = g + δg and f' = f + δf. Then, on the basis of (6), we obtain the variation of the functional

$$\delta J = (\delta g, \varphi_0^*) + \int_0^T (\delta f, \varphi^*)dt. \quad (7)$$

Hence, for evaluation of variations of functional J depending on different variations of data, it is not necessary to solve many problems of type (1):

$$\frac{\partial \varphi'}{\partial t} + A\varphi' = f', \ \varphi' = g' \quad \text{at} \quad t = 0 \quad (8)$$

with various f' and g'. It is enough to solve only one conjugate problem (2) and employ formula (7).

By means of formula (7) one can state the inverse problem of finding δg and δf for the set of functional δJ.

The above methods of the perturbation theory were based, to an extent, on the employment of conjugate equations of the theory of climate.

This is natural since the conjugate equations in this interpretation define the domain of influence of variations of input parameters, over the whole space, on the variations of the functional of the solution in a given region. Therefore the study of conjugate equations and understanding of general principles of climate modification on this basis is an important task.

However, this is not the only way to evaluate climate variations.

There is a simpler approach of direct integration of equations of climate. If the modification of climate proves to be essential because of the variation of input parameters then it is required to solve initial problems corresponding to different sets of input data.

We will analyze now ocean dynamics equations.

Let us turn to the component representation of the perturbation theory formulas. To this aim we consider the perturbed system of equations of atmospheric motions.

$$
\begin{gathered}
\frac{\partial \bar{\rho} u'}{\partial t} + \Lambda' u' - l\bar{\rho} v' - \bar{p}\,\frac{\partial \varphi'}{\partial x} - \mu\bar{\rho}\Delta u' = 0, \\
\frac{\partial \bar{\rho} v'}{\partial t} + \Lambda' v' - l\bar{\rho} u' - \bar{p}\,\frac{\partial \varphi'}{\partial y} - \mu\rho\Delta v' = 0, \\
g\bar{p}\vartheta' - \bar{p}\,\frac{\partial \varphi}{\partial z} = 0, \qquad (9) \\
\frac{\partial \bar{\rho} u'}{\partial x} + \frac{\partial \bar{\rho} v'}{\partial y} + \frac{\partial \bar{\rho} w'}{\partial z} = 0, \\
\frac{\partial \bar{\rho}\vartheta'}{\partial t} + \Lambda'\vartheta' + \frac{\gamma_a - \gamma}{\overline{T}}\,\bar{\rho} w' - \frac{\partial}{\partial z}\,\bar{\rho}\nu_1\,\frac{\partial \vartheta'}{\partial z} - \bar{\rho}\mu_1\,\Delta\vartheta' = 0
\end{gathered}
$$

with the boundary conditions

$$
\begin{gathered}
\frac{\partial \vartheta}{\partial z} = \alpha_s' \,(\vartheta' - \overline{\vartheta}'),\ \bar{\rho} w' = 0 \quad \text{at} \quad z = 0, \\
\qquad (10) \\
\frac{\partial \vartheta'}{\partial z} = 0,\ \bar{\rho} w' = 0 \quad \text{at} \quad z = H
\end{gathered}
$$

and the conditions of periodicity with respect to (x, y). Here $\vartheta' = \vartheta$

$+ \delta\vartheta'\vartheta' = \vartheta + \delta\vartheta$, ϑ and ϑ are climatic temperatures of the air at the level of the booth and the upper friction layer of the ocean, respectively, $\delta\vartheta$ and $\delta\vartheta$ are deviations from climatic values. We take

$$u' = u'_o, \quad v' = v'_o, \quad \vartheta' = \vartheta'_o \quad \text{at} \quad t = 0 \tag{11}$$

as initial data.

Now the conjugate system corresponding to the unperturbed state of the atmosphere is considered

$$-\frac{\partial\bar{\rho}u^*}{\partial t} - \Lambda u^* + l\bar{\rho}v^* - \bar{p}\frac{\partial\varphi^*}{\partial z} - \mu\bar{\rho}\Delta u^* = 0,$$

$$-\frac{\partial\bar{\rho}v^*}{\partial t} - \Lambda v^* - l\bar{\rho}u^* - \bar{p}\frac{\partial\varphi^*}{\partial y} - \mu\bar{\rho}\Delta v^* = 0,$$

$$g\bar{\rho}\vartheta^* - \bar{p}\frac{\partial\varphi^*}{\partial z} = 0, \tag{12}$$

$$\frac{\partial\bar{\rho}u^*}{\partial x} + \frac{\partial\bar{\rho}v^*}{\partial y} + \frac{\partial\bar{\rho}w^*}{\partial z} = 0,$$

$$-\frac{\partial\bar{\rho}\vartheta^*}{\partial t} - \Lambda\vartheta^* - \frac{\gamma_a - \gamma}{\bar{T}}\bar{\rho}w^* - \frac{\partial}{\partial z}\bar{\rho}\nu_1\frac{\partial\vartheta^*}{\partial z} - \bar{\rho}\mu_1\Delta\vartheta^* = 0$$

with the boundary conditions

$$\frac{\partial\vartheta^*}{\partial z} = \alpha_s\vartheta^*, \ \bar{\rho}w^* = 0 \quad \text{at} \quad z = 0,$$
$$\frac{\partial\vartheta^*}{\partial z} = 0, \ \bar{\rho}w^* = 0 \quad \text{at} \quad z = H \tag{13}$$

assuming the periodicity of solution with respect to (x, y) and the initial data

$$u = u^*_T, \; v^* = v^*_T, \; \vartheta^* = \vartheta^*_T \quad \text{at} \quad t = T, \tag{14}$$

where u^*_T, v^*_T and ϑ^*_T are functions to be defined below. Before constructing the formulas of the perturbation theory let us introduce the following notation:

$$\Lambda' = \Lambda + \delta\Lambda, \; \alpha'_s = \alpha_s + \delta\alpha_s$$

Repeating the above mentioned procedure we multiply the initial equation of system (9) by u^*, v^*, w^*, $RT\varphi^*$ and $\frac{g\overline{T}}{\gamma_a - \gamma}\vartheta^*$, respectively, and then sum up the results.

As a result of application of the perturbation theory we obtain the following important relation for the temperature variation:

$$\delta(\overline{\bar{\rho}\vartheta}^G_T) = -\int_{-\infty}^{T} dt \int_D (u^*\delta\Lambda u' + v^*\delta\Lambda v' + \frac{g\overline{T}}{\gamma_a - \gamma}\vartheta^* o\Lambda\vartheta')\, dD + q\int_{-\infty}^{T} dt \int_S [\alpha_s\delta\bar{\vartheta} + \delta\alpha_s(\bar{\vartheta}' - \vartheta')]\vartheta^* dS. \tag{15}$$

Assuming that the conjugate problem is solved at actual values of u, v, w then $\delta\Lambda = 0$ and we obtain

$$\delta(\overline{\bar{\rho}\vartheta}^G_T) = q\int_{-\infty}^{T} dt \int_S [\alpha_s\delta\bar{\vartheta} + \delta\alpha_s(\bar{\vartheta}' - \vartheta')]\vartheta^* dS. \tag{16}$$

The meaning of this formula is quite clear – the first term on the right (16) describes contribution into the temperature anomaly at the expense of the atmosphere – ocean interaction where

$$q\int_{-\infty}^{T} dt \int_S \alpha_s\delta\bar{\vartheta}\vartheta^* dS$$

makes allowance for the deviation of temperature of the surface friction layer from the climatic temperature, and the other term

$$q \int_{-\infty}^{T} dt \int_{S} \delta\alpha_s(\bar{\vartheta}' - \vartheta')\vartheta^* dS$$

describes the effects of deviation of the atmosphere – ocean heat transfer due to storms, non-standard dynamics of the ice cover, etc. It is evident from (16) that long-term temperature anomalies of large regions of the continents develop in the active layer of the ocean due to its interaction with the atmosphere.

Finally, if storms are neglected and the dynamics of ice is assumed known, the formula for forecasts of temperature anomalies has the simple form

$$\delta(\overline{\bar{\rho}\vartheta_T^G}) = q \int_{-\infty}^{T} dt \int_{S} \alpha_s \delta\bar{\vartheta}\vartheta^* dS. \qquad (17)$$

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