TURBULENCE AND DIFFUSION IN STRATIFIED FLOWS

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RESUMEN

En este trabajo se presentan: Criterios de estabilidad hidrodinámica para flujos estratificados. La teoría de similitud para la turbulencia en flujos estratificados cizallantes. Casos especiales de convección cizallante, convección libre y formación de capas bajo estratificación muy estable. La forma del tensor de difusividad del vórtice. Difusión longitudinal en flujos cizallantes. Descripción lagrangeana de la difusión turbulenta.

ABSTRACT

Criteria of hydrodynamic stability for stratified flows. The similarity theory for turbulence in shear stratified flows. Special cases of shear convection, free convection and layering under very stable stratification. The form of the eddy diffusivity tensor. Longitudinal diffusion in shear flows. Lagrangean description of turbulent diffusion.

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STABILITY CRITERIA

A stratification of an inhomogeneous fluid or a gas is a change of its density along vertical, the direction of gravity force. In a stratified fluid, vertical displacements of fluid particles from their equilibrium positions produce the *buoyancy* forces $(\rho_1 - \rho)\, g$ where ρ_1 is the density of the fluid particles, ρ is the density of the surrounding medium and g is the acceleration of gravity. The work of *buoyancy* forces generates an exchange between the potential energy of stratified fluid in the gravity field and the kinetic energy of fluid motions.

To determine the direction of this energy exchange it is convenient to introduce a concept of the *potential density* ρ_*. For fluids determined by only two independent thermodynamical parameters (say, by pressure p and temperature T, for instance, dry air or fresh water) ρ_* is the density of fluid particle after isentropic transformation to a standard pressure. Let z be a vertical co-ordinate increasing upwards. If $\frac{\partial \rho_*}{\partial z} < 0$ then the work of buoyancy forces transforms the kinetic energy into the potential energy; the stratification in this case is called *stable*. If $\frac{\partial \rho_*}{\partial z} > 0$ then the work of buoyancy forces transforms part of the potential energy into kinetic energy of motions, i.e. it generates *convection*; the stratification in this case is called *unstable*. For fluids mentioned above with the equation of state of the form $\rho = \rho\,(p, \mathrm{T})$ the vertical gradient of potential density is proportional to and has the sign of

$$\frac{\partial \rho}{\partial z} + \frac{\rho g}{C^2} = -\alpha\rho\left(\frac{\partial T}{\partial z} + \frac{\alpha T g}{C_p}\right) = -\frac{\alpha T g}{C_p}\frac{\partial \eta}{\partial z} \qquad (1)$$

if the hydrostatic equation $\frac{\partial p}{\partial z} = -\rho g$ is valid; here C is the speed of sound, α is the thermal expansion coefficient (for ideal gases $\alpha = \frac{1}{T}$),

Cp is the specific heat capacity under constant pressure, η is the entropy. The equation (1) enables us to express stability criterium in terms of density or temperature or entropy vertical gradients (if a fluid is determined by one more thermodynamical parameter, for instance, moist air --by specific humidity, sea water– by salinity, then the definition of ρ_* and the condition (1) are more complicated; thus for the sea water the definition of ρ_* includes both isentropic *and isohaline* processes).

In the presence of shear flow in a stratified fluid (mean shear in the case of turbulent flow) the hydrodynamical stability diminishes. Further on we shall restrict ourselves to the case of horizontal stationary parallel flows with a vertical gradient $\frac{\partial u}{\partial z}$ of velocity u, choosing the axis $x_1 = x$ of Cartesian co-ordinates x_i along the flow and the axis $x_2 = y$ laterally in horizontal plane and keeping the notation $x_3 = z$ for the vertical co-ordinate. Turbulent flow of this kind will be considered as *statistically* stationary, horizontally homogeneous (but non-isotropic and vertically inhomogeneous), locally-homogeneous and, finally, locally-axisymmetric relative to vertical direction (in particular, locally-isotropic in horizontal planes). All of mean one-point characteristics of such a flow may depend on the height z only. This model is valid for instance for lower parts of the atmospheric boundary layer (Monin, 1965) and for upper parts of the oceanic boundary layer where the Earth's rotation effects are negligible.

In flows of this kind hydrodynamic instability develops (generating finally turbulence) not only under unstable but also under stable stratification while the Richardson criterium is fulfilled,

$$R_i = -\frac{g}{\rho}\frac{\partial \rho_*}{\partial z}\left(\frac{\partial u}{\partial z}\right)^{-2} < R_{i_{cr}} \tag{2}$$

where $R_{i_{cr}}$ is the critical value (positive) of the Richardson number R_i. A similar criterium for turbulent flows is convenient to establish requiring that the loss of turbulent kinetic energy due to the work of buoyancy forces under stable stratification, $gM, M = \overline{\rho' w'}$, be less than the rate of turbulent kinetic energy generation due to the work of Reynolds stresses on the mean shear, $\rho u_*^2 \frac{\partial \bar{u}}{\partial z}$, $u_*^2 = -\overline{u'w'}$ (here and further on we use the notations $u_1 = u$, $u_2 = \nu$, $u_3 = w$ for Cartesian velocity components and denote mean values by bar and turbulent fluctuations by primes; M is the vertical turbulent mass flux; ρu_*^2 is a Reynolds stress; u_* is called a *friction velocity*). This criterium has the form

$$Rf = \frac{g}{\rho} M \left(u_*^2 \frac{\partial \bar{u}}{\partial z} \right)^{-1} < R \tag{3}$$

where Rf is the so-called flux Richardson number and R is its critical value (which is positive and has the order of 0.1 according to empirical data). The connection between the criteria (2) and (3) may be established if we introduce the exchange coefficients for momentum $K_u = u_*^2 \left(\frac{\partial \bar{u}}{\partial z} \right)^{-1}$ and for mass $K\rho = -M \left(\frac{\partial \rho_*}{\partial z} \right)^{-1}$; then $Rf = (K\rho / K_u) Ri$.

Let us note that if the density changes $\delta\rho$ are produced mainly by temperature changes δT, i.e., $\frac{\delta\rho}{\rho} \approx -\alpha\delta T$, then $M \approx -\frac{\alpha}{Cp} H$ where $H = C_p \rho \overline{T'w'}$ is the vertical turbulent heat flux. Besides, $\frac{1}{\rho}\frac{\partial \rho_*}{\partial z} \approx -\alpha \frac{\partial \theta}{\partial z}$ where θ is the *potential* temperature, i.e. the temperature of a fluid particle after adiabatic transformation to a standard pressure. Further on we shall restrict ourselves to this case and use the notation $\beta = \alpha g$ for the buoyancy parameter.

Under a stable stratification $Ri > 0$, $Rf > 0$, $M > 0$, $H < 0$ and the energy loss due to work against buoyancy forces leads to a weakening of turbulence (and if Rf > R – even to the impossibility of its existence). On the contrary, under unstable stratification Ri < 0, Rf < 0, M < 0, H > 0 and the generation of kinetic energy due to work of buoyancy forces should lead to a strengthening of turbulence. The same is true of course for turbulent diffusion: it should decelerate under stable and accelerate under unstable stratification. Main knowledge on turbulent diffusion may be found in early Monin's reviews (1956, 1959), in Chapter V of Monin and Yaglom's book (1971) and in Pasquill's book (1962).

SIMILARITY THEORY

The similarity theory developed by Obukhov and Monin in 1946-54 proved to be convenient for description of the dependence of turbulence upon stratification (see its presentation in Chapter IV of Monin and Yaglom, 1971). According to this theory, all statistical characteristics of turbulent flows defined above which are determined by turbulent eddies of not too small scales (for which molecular viscosity and heat conduction effects are negligible) may depend upon but three "external parameters": the vertical turbulent momentum flux or the Reynolds stress $\tau = -\overline{\rho u'w}$ (actually upon the friction velocity $u_* = (\tau/\rho)^{1/2}$), the vertical turbulent heat flux H (actually upon $\frac{H}{C_p\rho}$) and the buoyancy parameter $\beta = \alpha g$. One can construct the scales of length L, of velocity u_* and of temperature T_* by means of these three parameters, namely

$$L = -\frac{u_*^3}{æ\,\frac{H}{C_p\beta}}; \quad T_* = -\frac{1}{æ u_*}\,\frac{H}{Cp\rho} \qquad (4)$$

(where æ ≈ 0,4 is the von Karman constant introduced for the sake of convenience). The main assumption of the similarity theory is that all statistical characteristics of velocity and temperature field (with the restriction mentioned above) measured by the scales u_* and T_* are some universal functions of dimensionless heights $\zeta = \frac{z}{L}$ of observation points.

Applying this assumption to the mean vertical gradients of velocity $\frac{\partial \bar{u}}{\partial z}$ and of potential temperature $\frac{\partial \theta}{\partial z}$ (note that potential temperature θ and absolute temperature T are practically indistinguishable in thin layers of a fluid where vertical pressure differences are small, for instance in a surface layer of the atmosphere some tens of metres thick) we get

$$\frac{\partial \bar{u}}{\partial z} = \frac{u_*}{æL} \varphi\ (\zeta);\ \frac{\partial \bar{\theta}}{\partial z} = \frac{T_*}{L} \varphi_1\ (\zeta) \qquad (5)$$

where $\varphi\ (\zeta)$ and $\varphi_1\ (\zeta)$ are some universal functions over the range $-\infty < \zeta < \infty$ while the limit in $H \to 0$ (where $/L/ \to \infty$ and $\zeta \to 0$) corresponds to the case of neutral stratification, the range $\zeta < 0$ (while $H > 0$) – to unstable stratifications, and the range $\zeta > 0$ (while $H < 0$) – to stable stratifications. We get $Rf = \frac{1}{\varphi(\zeta)}$ from (5), and if $\frac{K_T}{K_u} = \gamma\ (\zeta)$ then $Ri = \frac{1}{\gamma(\zeta)\varphi(\zeta)} = \frac{\varphi_1(\zeta)}{\varphi_2(\zeta)}$, and therefore $\varphi_1(\zeta) = \varphi(\zeta)/\gamma(\zeta)$.

Under neutral stratification ($H \to 0$) the velocity gradient $\frac{\partial \bar{u}}{\partial z}$ should be finite and therefore $\varphi(\zeta) \sim \frac{1}{\zeta}$ when $\zeta \to 0$. It is natural to expect in this case that the limit $\gamma(0) = \gamma_0$ should be finite (near to one according to measurements) and therefore $\varphi_1\ (\zeta) \sim \frac{1}{\gamma_0 \zeta}\cdot$ In the limit in this case we get from (5)

$$\frac{\partial \bar{u}}{\partial z} = \frac{u_*}{æz}\ ; \ \frac{\partial \bar{\theta}}{\partial z} = \frac{T_*}{\gamma_0 z} \qquad (6)$$

and therefore the mean vertical velocity and temperature profiles under a near-neutral stratification are described by the well-known logarithmic equations.

One can get the asymptotics for a very unstable stratification ($H \to \infty$, $\zeta \to -\infty$ putting $u_* \to 0$ under fixed values of remaining parameters, i.e., considering the case of the *free convection* (in the absence of mean flow). The temperature gradient $\frac{\partial \bar{\theta}}{\partial z}$ should be finite in this limit and therefore the function $\varphi_1\ (\zeta)$ should be asymptotically proportional to $\zeta^{-4/3}$. In this case the limit $\gamma\ (-\infty) = \gamma_1$ may be expected to be finite (γ_1 is nearly 3 according to measurements of Australian scientists) and therefore the function $\varphi\ (\zeta)$ should also be asymptotically proportional to $\zeta^{-4/3}$. The equations (5) in this limit take the form

$$\frac{\partial \bar{u}}{\partial z} = C\gamma_1 \frac{\beta H}{C_p \rho}^{-1/3} u_*^2 \ z^{-4/3}\ ; \ \frac{\partial \bar{\theta}}{\partial z} = -C \left(\frac{H}{C_p \rho}\right)^{2/3} \beta^{-1/3} z^{-4/3} \qquad (7)$$

where C is a numerical constant.

In the case of a very stable stratification ($\zeta \to \infty$) the flux Richardson number Rf tends to its critical value R from below and therefore $\varphi\ (\zeta) \sim \frac{1}{R}$ and the equations (5) take the form

$$\frac{\partial \bar{u}}{\partial z} = \frac{u_*}{æRL}\ ; \ \frac{\partial \bar{\theta}}{\partial z} = \frac{T_*}{RL\gamma(\zeta)} \qquad (8)$$

Therefore the velocity profile should be nearly linear under a very stable stratification. All the available data show that the function $\gamma(\zeta)$ has very small values in this case. In the limiting case of stable stratification (which is a discontinuity surface with a density jump) γ tends to zero because the mass and heat turbulent transfer through such a surface is impossible while the momentum transfer is still possible due to pressure fluctuations. In the R. Long theory (1970) supported by Monin, Neyman and Filyushkin (1970) empirical data on density profiles in great oceanic depths, $\gamma(\zeta)$ diminishes as ζ^{-2} under a very stable stratification. The case of a very stable stratification will be discussed in more detail below.

Let us apply the similarity theory to a velocity and temperature turbulent fluctuations in a given space point. According to the similarity theory the probability density for these fluctuations should take the form

$$p(u', v', w', T') = \frac{1}{u_*^3 |T_*|} f\left(\frac{u'}{u_*}, \frac{v'}{u_*}, \frac{w'}{u_*}, \frac{T'}{|T_*|}, \zeta\right) \tag{9}$$

where f is some universal function of five variables whose behaviour under $\zeta \to 0, \pm\infty$ may be established similarly to above (see Chapter IV in Monin and Yaglom, 1971). Three out of 10 second order moments of u', v', w', T' – fluctuations vanish due to symmetry properties of the flow (let us emphasize however that the longitudinal heat flux $H_1 = C_p \rho \overline{u'T'}$ has no reason to vanish); two are constants $(\overline{u'w'} = -u_*^2$ and $\overline{T'w'} = \frac{H}{C_p \rho})$ and five remaining moments in dimensionless form $\frac{\sigma u}{u_*}, \frac{\sigma v}{u_*}, \frac{\sigma w}{u_*}, \frac{\sigma_T}{|T_*|}$ and (σ means a mean square value) are universal functions of ζ, the behaviour of which under $\zeta \to 0, \pm\infty$ may be established as above. The semi-empirical calculation of all these functions may be found in Monin (1965).

Let us consider in more detail the less known quantity H_1. It is clear that under neutral stratification and in the case of free convection it vanishes and otherwise has a sign opposite to the vertical heat flux H (beause u′ and w′ are negatively correlated). Measurements in the surface layer of the atmosphere by Zubkovsky and Tsvang (1966) supported by other authors have shown that in a near-neutral stratification $-\frac{H_1}{H} \gtrsim 3$. This ratio diminishes under growing instability, its usual values in the atmosphere being in the neighbourhood of 1-2; its values seem to grow rapidly when stability increases (Yaglom, 1969).

Note finally that the exchange coefficient for momentum is equal to

$$K_u = \frac{æ u_* L}{\varphi(\zeta)} = æ u_* L \cdot Rf \qquad (10)$$

Under neutral stratification and under any stratification when $z \ll /L/$ this coefficient is equal to $æ u_* z$; under stable stratification it tends to the constant $æ u_* LR$ when ζ increases; under unstable stratification it is asymptotically proportional to $z^{4/3}$. The exchange coefficient for heat is equal to K_u times $\gamma(\zeta)$.

The predictions of the similarity theory have got a great deal of support in numerous atmospheric and laboratory measurements reviewed for instance in the book (Monin and Yaglom, 1971).

SHEAR CONVECTION

Silitinkevich (1971) noted that in a shear flow under unstable stratification there was a layer with not too small and not too large values of ζ (say, $-1 < \zeta < -0.1$) where the longitudinal velocity fluctuations u′ get the energy mainly from the mean flow due to work of Reynolds stresses while the vertical fluctuations w′ get the energy mainly from the potential energy of stratification due to work of buoyancy forces, the energy exchange between longitudinal and vertical fluctuations being comparatively small.

One may expect that in this layer the hydrodynamical equations are invariant under affine transformations of co-ordinates with different coefficients for horizontal and vertical directions (Betchow and Yaglom, 1971). In this case it is reasonable to prescribe different dimensions L_h and L_v to horizontal and vertical scales. Then the height z has the dimension L_v, the buoyancy parameter β – dimension $L_v\, t^{-2}\, T^{-1}$ where t and T are the dimensions of time and temperature, the friction velocity u_* – dimension $L_h^{1/2}\, L_v^{1/2}\, t^{-1}$, the parameter $\frac{H}{C_p \rho}$ – dimension $L_v\, t^{-1}\, T$. There are no constant scales of length, velocity and temperature in this case, the only scales of vertical and temperature being

$$W = \left(\frac{\beta H z}{C_p \rho}\right)^{1/3}; \ \theta = \left(\frac{H}{C_p \rho}\right)^{2/3} (\beta z)^{-1/3} \tag{11}$$

while the scale of horizontal velocity is equal to u_*^2/W. The equations (7) are valid in this case but the probability density (9) takes the form

$$\rho(u', v', w', T') = \frac{W}{u_*^4 \theta} f \left(\frac{Wu'}{u_*^2}, \frac{Wv'}{u_*^2}, \frac{w'}{W}, \frac{T'}{\theta}\right) \tag{12}$$

One can find in (Zilitinkevich, 1971) experimental data on the moments $\overline{u'T'}$ and $\overline{u'w'T'}$ in favour of (Ludlam, 1967). At the same time in the region of a free convection $\zeta \ll -1$ all the components of velocity fluctuations have the scale W. Thus for instance σ_u is proportional to $\frac{u_*^2}{W} \sim z^{-1/3}$ in the layer of shear convection, i.e. it diminishes with height, while in the layer of free convection it increases as $W \sim z^{1/3}$; on the contrary σ_w increases as $z^{1/3}$ everywhere.

LAYERING UNDER STABLE STRATIFICATION

Recent measurements in the ocean by means of fast-response temperature, conductivity and sound velocity soundings (the latter two quantities depend very strongly on temperature and somewhat weaker on salinity and pressure) showed that the ocean whose stratification is usually very stable consisted almost everywhere of quasi-homogeneous layers with thicknesses from tens of metres to metres and less divided by very thin sheets with jumps of thermodynamical parameters ("steps" on vertical profiles of the parameters).

Typical examples are shown in Figs. 1-3 obtained on the Inst. Oceanology, Ac. Sci. USSR research vessels (in the tropical zone of the Indian Ocean during winter monsoon). Fig. 1 shows the conductivity profile in the upper 1 200 m of the ocean containing numerous "steps" especially below the thermocline (including inversion layers where conductivity increases with depth). Fig. 2 shows three sound velocity profiles in the uppermost layer of the ocean several tens of metres thick (which usually is considered as a homogeneous one) obtained at the same station with time intervals of 10 min. There are thin sublayers here well reproducible by repeated soundings. Large time-span of thin layers is demonstrated again in Fig. 3 where 6 pairs of temperature and salinity profiles at depths of 650 to 850 m are shown which were obtained with time intervals of 30 min. Similar layering was discovered by radar measurements in stable stratified layers of the upper atmosphere (see Ludlam's review, 1967).

The layering itself and the long time-span of individual thin layers means that turbulence under these conditions doesn't fill in all the space continuously but is concentrated inside the quasi-homogeneous layers and it is so weak that it cannot penetrate the sheets dividing the layers. The Reynolds numbers of such a turbulence defined by thicknesses of the layers are small (seem to be of the order of 10^4-10^5); thus the turbulence is underdeveloped and one doesn't expect the inertia range of spectrum to exist which is predicted by Kolmogoroff theory for turbulence with very large Reynolds numbers. Finally, this turbulence is a *local* one, i.e. it is determined by properties of the layer but not by the depth of its whereabouts.

One may think that such a layering is a typical property of turbulence in a very stably stratified fluid. The generation mechanism for thin layers is now under discussion. The most likely ones are Woods and Wiley (1972) idea of shear instability of internal waves (in regions where $Ri < \frac{1}{4}$) and Stommel and Fedorov's (1967) idea of "lateral convection" under conditions of horizontally inhomogeneous stratification.

EDDY DIFFUSIVITY TENSOR

In the Eulerian framework the description of the concentration field c of a conservative matter diffusing in a turbulent flow is based on the averaged molecular diffusion equation

$$\frac{\partial \bar{c}}{\partial t} + u_\alpha \frac{\partial \bar{c}}{\partial x_\alpha} = D \, \Delta \, c - \frac{\partial}{\partial x_\alpha} \overline{c' u'_\alpha} \qquad (13)$$

where D is the molecular diffusivity and $c'u'_\alpha$ is the turbulent flux of the matter; the summation rule over repeated Greek indices is supposed here and further on. When the diffusion time is small the interaction between molecular and turbulent diffusion may be important – molecular diffusion accelerates the turbulent one (Chapter V of Monin and Yaglom, 1971), but in large diffusion time the term D Δc in the right hand side of (13) may be neglected as compared to the second term. In a large diffusion time the Boussineq hypothesis on linear dependence of the turbulent flux of matter upon the mean gradient of its concentration is applicable, i.e.

$$\overline{c' u'_\alpha} = - K_{\alpha\beta} \frac{\partial \bar{c}}{\partial x_\beta} \qquad (14)$$

The eddy diffusivity tensor $K_{\alpha\beta}$ introduced here is to be considered as a mean characteristic of the turbulence field not depending on the shape of the concentration field. Note that there is no reason for the symmetry of the tensor in the indices α and β. In turbulent flows considered here $K_{xy} = K_{yx} = K_{zy} = K_{yz} = 0$ due to their symmetry; other components of the tensor $K_{\alpha\beta}$ generally have non-zero values and may depend but upon the height z. Thus the semiempirical turbulent diffusion equation for these flows has the form

$$\frac{\partial \bar{c}}{\partial t} + \bar{u}(z) \frac{\partial \bar{c}}{\partial x} = \left[K_{xx}(z) \frac{\partial^2 \bar{c}}{\partial x^2} + K_{xz}(z) \frac{\partial^2 \bar{c}}{\partial x \partial z} \right] + \\ + K_{yy}(z) \frac{\partial^2 \bar{c}}{\partial y^2} + \frac{\partial}{\partial z} \left[K_{zx}(z) \frac{\partial \bar{c}}{\partial x} + K_{zz}(z) \frac{\partial \bar{c}}{\partial z} \right] \tag{15}$$

Let us emphasize that there is no reason to neglect the terms with the non-diagonal components K_{xz} and K_{zx} of the tensor $K_{\alpha\beta}$ (Yaglom, 1969, 1972). In flows considered here

$$K_{xz} = - \frac{H_1}{C_p \rho} \left(\frac{\partial \theta}{\partial z} \right)^{-1} = \frac{H_1}{H} K_{zz} < 0 \tag{16}$$

and under neutral stratification $|K_{xz}|$ is more than three times larger than K_{zz}. Considering the flow with $\frac{\partial \bar{u}}{\partial z} > 0$ and with an initial concentration field $\bar{c} = \bar{c}(x)$, $\frac{\partial \bar{c}}{\partial x} < 0$, we'll have $\overline{u'c'} > 0$ and $\overline{w'c'} < 0$

H

(because of negative correlation between u' and w'); therefore

$$K_{zx} = - \overline{w'c'} \left(\frac{\partial \bar{c}}{\partial x} \right)^{-1} = \frac{\overline{w'c'}}{\overline{u'c'}} K_{xx} < 0 \tag{17}$$

and under neutral stratification $|K_{zx}|$ is approximately three times less than K_{xx}. Note finally that different components $K_{\alpha\beta}$ may depend on the height differently; for instance $K_{xx} \sim K_{yy} \sim \frac{u_*^4 z}{W^3} =$ const, $K_{xz} \sim K_{zx} \sim \frac{u_*^2 z}{W} \sim z^{2/3}$, $K_{zz} \sim W_z \sim z^{4/3}$ in the layer of shear convection.

Over very large diffusion distances a horizontal spreading of matter is produced by both small-scale turbulence described by the similarity theory and large-scale eddies with vertical axes which obey quite different regularities. Horizontal spreading due to small-scale turbulence may be neglected in many problems. If large-scale eddies belong to the inertia range of turbulence spectrum, i.e. they are determined by the only parameter – the rate of spectral energy transfer ϵ, then horizontal size of a diffusing cluster increases in time as $(\epsilon t^3)^{1/2}$. Joint action of horizontal and vertical diffusion in this case was calculated by Monin (1969).

LONGITUDINAL DIFFUSION IN A SHEAR FLOW

It is natural to expect (and it gets a support in measurements) that $\sigma_H > \sigma_v$ and therefore $K_{xx} > K_{yy}$ because longitudinal velocity fluctuations u′ feed directly on the energy of a mean flow and lateral horizontal fluctuations v′ – on the energy of longitudinal fluctuations. Therefore a diffusing cluster originated by an instantaneous point source should be elongated in the direction of the flow but the ratio of its horizontal axes should not change in time.

However actually observed clusters grow along the flow faster than in lateral direction and become more and more elongated in the flow direction. This effect may be explained by an interaction between lateral diffusion and a shear. Such an interaction is convenient to demonstrate in a simple model of the flow with constant shear Γ (i.e. $\bar{u}(z) = \Gamma z$) and constant diffusivities. The solution of the diffusion equation (15) in an infinite space corresponding to an instantaneous point source at a moment $t = 0$ in the origin is in this case the Gaussian function $\bar{c}\,(x - \Gamma zt, y, z, t)$ with second moments

$$D_{xx} = 2K_{xx}t + \Gamma(K_{xz} + K_{zx})t^2 + \frac{2}{3}\Gamma^2 K_{zz}\, t^3;$$

$$D_{yy} = 2K_{yy}t;\ D_{zz} = 2K_{zz}t;\ D_{xy} = D_{yz} = 0; \qquad (18)$$

$$D_{xz} = (K_{xz} + K_{zx})t + \Gamma K_{zz}t^2$$

(one may derive these equations multiplying (15) by x^2, y^2, z^2, xz and integrating over all the space). It is seen that due to the term with the factor $\Gamma^2 K_{zz}$ the longitudinal variance D_{xx} increases in time much faster than lateral variances; in large diffusion time the longitudinal size of a cluster increases as $\Gamma t \sqrt{K_{zz}t}$ while lateral sizes – as $\sqrt{K_{yy}t}$ and $\sqrt{K_{zz}t}$.

A mean shear of actual flows changes in space and the calculation of longitudinal diffusion becomes more complicated. For flows in pipes such a calculation was undertaken by Taylor (1953, 1954) who showed that the longitudinal distribution of matter from an instantaneous point source was rapidly becoming the Gaussian one with the variance 2Kt while the effective coefficient of longitudinal diffusion K was equal to $\frac{R^2 U^2}{48D}$ in laminar flows (where R is the pipe radius, U is the mean velocity and D is the molecular diffusivity) and equal to 10,1 Ru_* in turbulent flows (where u_* is the friction velocity on the wall). In boundary layers with a velocity profile $\bar{u}(z)$ determined by the equation (5) the exact analytical solution of the diffusion equation (15) is difficult and it is profitable to use Lagrangean methods here.

LAGRANGEAN DESCRIPTION OF TURBULENT DIFFUSION

Lagrangean description of diffusion is a calculation of co-ordinates X(t), Y(t), Z(t) of the diffusing particle situated in the initial moment $t = 0$, say, in the point with the co-ordinates $x = y = 0$, $z = h$ and particle velocity components U(t), W(t) which are the time-derivatives of the co-ordinates.

From the point of view of similarity theory these functions may depend in addition to their argument t also upon the external parameters u_*, $\frac{H}{C_p \rho}$ and β, on the initial height h of the diffusing particle and on the roughness parameter z_0 of the underlying surface which determines the velocity $\bar{u}(z)$ according to the equation $\bar{u}(z) = \frac{u_*}{\partial e}\left[f\left(\frac{z}{L}\right) - f\left(\frac{z_0}{L}\right)\right]$ obtained by integration of the first equation (5) (where $f(\zeta)$ is the integral of $\varphi(\zeta)$).

It is clear however that the influence of the parameter should be decreasing in time and be neglected when $t \gg \frac{h}{u_*}$ (and when $t \gg \frac{z_0}{u_*}$ if $h \lesssim z_0$); further on we'll consider only such large times. Then the parameter z_0 would influence statistical characteristics of particle velocities only through an additive constant $-\frac{u_*}{æ} f\left(\frac{z_0}{L}\right)$ in the expression for U(t). Applying such a similarity theory to mathematical expectations of Lagrangean velocities we get

$$\overline{U(t)} = \frac{u_*}{æ}\left[F\left(\frac{u_* t}{L}\right) - f\left(\frac{z_0}{L}\right)\right]; \quad \overline{W(t)} = u_* F_1\left(\frac{u_* t}{L}\right) \qquad (19)$$

where $F(\tau)$ and $F_1(\tau)$ are some universal functions, while $\overline{V(t)} = 0$

due to symmetry of the flow. The functions F(t) and $F_1(t)$ may be obtained by integration of (19) over t. Asymptotic behaviour of the functions X(t) and Z(t) in the limits of neutral, very stable and very unstable stratification may be established similarly to the case of Eulerian description (Monin and Yaglom, 1971). For instance, under neutral stratification the parameter L should be cancelled out of the equations (19) and therefore the functions $F(\tau)$ and $f(\zeta)$ should be logarithmic and the function $F_1(\tau)$ should be a constant, i.e.

$$\overline{U(t)} = \frac{u_*}{æ} \ln \frac{cu_* t}{z_0}; \quad \overline{W(t)} = bu_*;$$

$$\overline{X(t)} \approx \frac{u_* t}{æ} \ln \frac{cu_* t}{ez_0}; \quad \overline{Z(t)} \approx bu_* t \tag{20}$$

where b and c are some numerical constants and e is the bassis of natural logarithms. This case was studied by Ellison (1959) and Batchelor (1964) and the generalization for any stratification – by Gifford (1962) and Yaglom (1965). According to Yaglom (1965) the constant b seems to be in the range 0.3-0.4 and the constant c is approximately three times less.

According to the similarity theory the probability density of diffusing particle co-ordinates (and concentration of matter from an instantaneous point source proportional to it) in large diffusion times should have the form

$$p(X, Y, Z) = (\overline{Z})^{-3} \, f\left(\frac{X - \overline{X}}{\overline{Z}}, \frac{Y}{\overline{Z}}, \frac{Z}{\overline{Z}}; \frac{\overline{Z}}{L}\right) \tag{21}$$

where f is some universal function. Under neutral stratification and in the case of free convection the fourth argument $\overline{Z}/L$ is insignificant and in other cases its influence seems to be rather weak and perhaps may be neglected as a first approximation. Central moments of the order n of the distribution (21) are then proportional to $(\bar{Z})^n$. In particular the dispersion tensor $D_{ij}(t)$ has the form $C_{ij}[\bar{Z}(t)]^n$. It is true for longitudinal dispersion as well; under neutral stratification it is proportional to t^2 and in the case of free convection – to t^3.

The concentration distribution downstream from a stationary point source $\bar{c}(X, 0, 0)$ may be obtained by means of integration of $p(X, 0, 0)$ over all values of t; it is approximately proportional to $(\bar{Z}^2 U)^{-1}_{\bar{X}=X}$. Under neutral stratification it decreases with the distance from the source a little less than X^{-2} and in the case of free convection – as X^{-3}. For the linear stationary source on the Y-axis an additional integration over Y is needed. The concentration $\bar{c}(X, 0)$ in this case is approximately proportional to $(\bar{Z}\bar{U})^{-1}$; under neutral stratification it decreases as X^{-1} and in the case of free convection – as $X^{-3/2}$ (Monin and Yaglom, 1971) (Gifford, 1962).

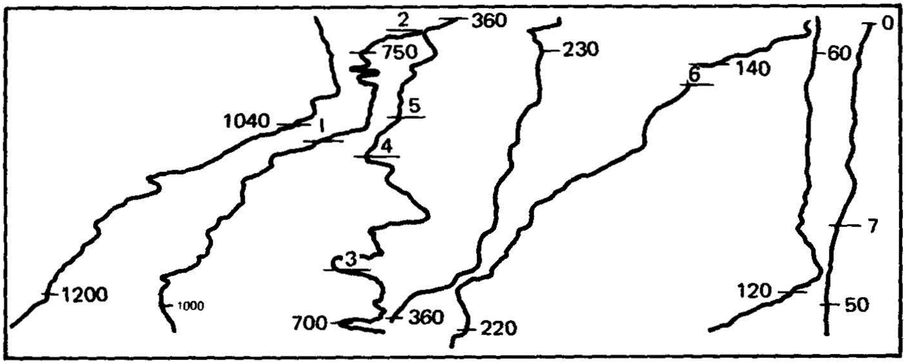

Figura 1. The conductivity profile in the upper 1 200 m layer of water in the tropical zone of the Indian Ocean during winter monsoon. Horizontal scale is the conductivity in arbitrary units (increases to the right). For convenience the profile is cut into 7 parts. Depths in metres are shown by figures.

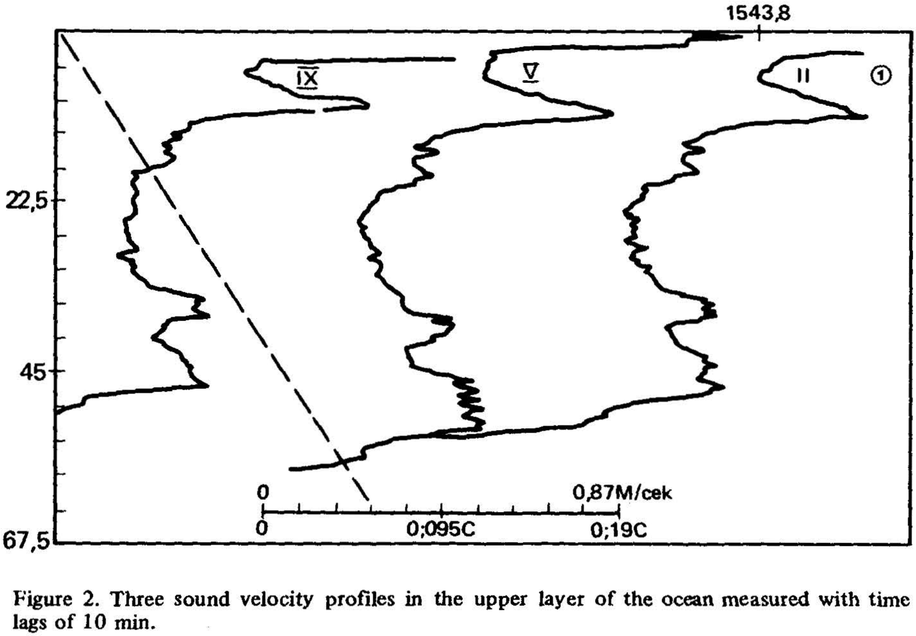

Figure 2. Three sound velocity profiles in the upper layer of the ocean measured with time lags of 10 min.

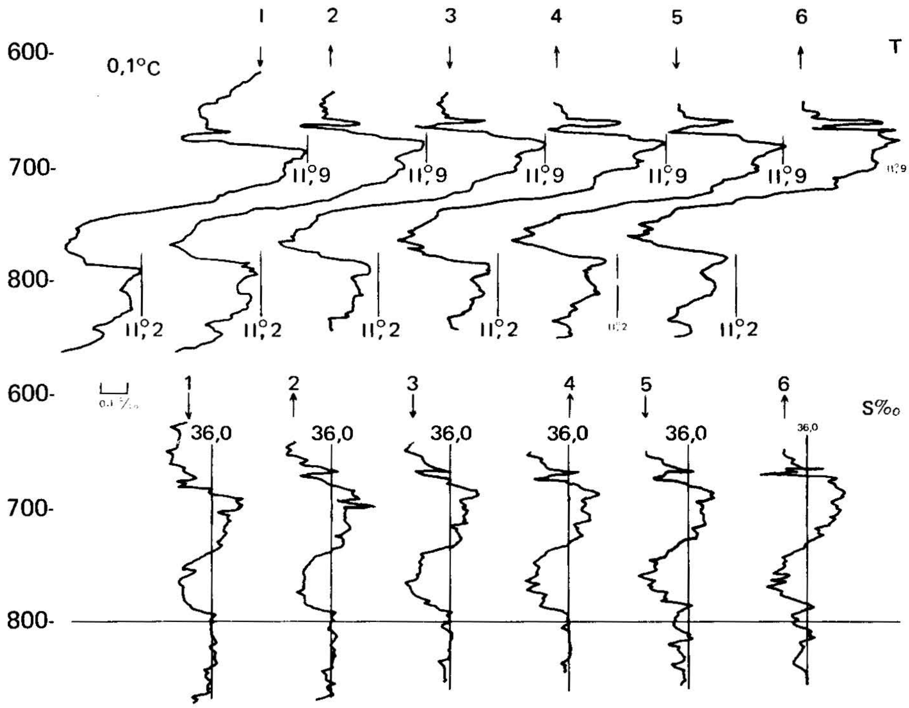

Figure 3. Six pairs of temperature and salinity profiles at depths of 650 to 850 m in Timor Sea measured with time lags of 30 min.

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