FURTHER STUDIES ON THE THERMODYNAMIC PREDICTION OF OCEAN TEMPERATURES

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RESUMEN

Se obtiene una solución en serie para la ecuación de conservación de energía térmica, aplicada a la capa superior de los océanos.

Dados los valores iniciales de las anomalías de la temperatura en la superficie de los océanos, la solución da los valores para un tiempo posterior, como funciones de la escala o tamaño de las mismas anomalías iniciales, así como de la velocidad del viento superficial, de la velocidad de las corrientes oceánicas y del coeficiente de difusión horizontal turbulenta.

La solución se aplica para determinar el error de truncamiento cometido cuando se aproxima la derivada, respecto al tiempo, de la anomalía de la temperatura por la fórmula de diferencias finitas de Euler y se usa un intervalo de tiempo hacia adelante. Dado un límite para el error de truncamiento, el lapso de tiempo permisible depende de la escala o tamaño de las anomalías iniciales, la velocidad del viento superficial, la velocidad de las corrientes oceánicas y de la difusión horizontal turbulenta.

Para el caso en que no se incluyen los términos debidos a la advección de calor por corrientes oceánicas medias e intercambio horizontal turbulento, la solución es independiente del tamaño de las anomalías, y un intervalo de tiempo de 30 días parece justificarse. Sin embargo, cuando estos términos de transporte horizontal se incluyen, se debe usar un intervalo de tiempo más pequeño, que depende de los parámetros arriba mencionados.

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ABSTRACT

A series solution for the conservation of thermal energy equation applied to the upper layer of the oceans is obtained.

Given the initial values of the surface ocean temperature anomalies, the solution yields the values at a later time as functions of the scale, or size, of the initial anomalies themselves, as well as of the surface wind speed, the ocean current speed and the horizontal mixing coefficient.

The solution is applied to determine the truncation error made when one approximates the time derivative of the temperature anomaly by the finite-difference Euler approximation, and uses a forward time step. Given an upper limit for the truncation error, the permissible time step depends on the scale or size of the initial anomalies, the surface wind speed, the ocean current speed and the horizontal mixing.

For the case when the terms due to advection of heat by mean ocean currents and horizontal mixing are neglected, the solution is independent of the size of the anomalies, and a time step of 30 days appears to be justified. However, when these horizontal transport terms are included, a shorter time step must be used which depends on the above mentioned parameters.

1. INTRODUCTION

In a recent paper (Adem, 1970) a thermodynamic approach to surface ocean temperature prediction was presented, and applied with some success to make predictions of the month-to-month changes of ocean temperature in the Atlantic and Pacific Oceans in the Northern Hemisphere.

The equation for conservation of thermal energy in the upper layer of the oceans is the forecasting equation used, and the predictions are obtained with one single time step of the order of a month. This appears to be a very long time step as compared with those used in numerical weather prediction, where the permissible time steps are of the order of minutes and at most of hours.

It is the main purpose of this investigation to explore the nature of the solution of the equations used in the thermodynamic model, and how the permissible time step depends on the temperature anomaly scale that one wants to predict, as well as on the diffusion coefficient, the mean ocean currents and on other factors.

2. BRIEF DESCRIPTION OF THE EQUATION

The equation derived in the previous paper (Adem, 1970) is the following:

$$\frac{\partial T_S}{\partial t} = - \mathbf{v}_{S_T} \cdot \nabla T_S + K_S \nabla^2 T_S + \frac{1}{h \rho_S C_S} (E_S - G_2 - G_3) - \frac{W}{h} \tag{1}$$

where ∇ is the two-dimensional horizontal gradient operator; T_S, the surface ocean temperature; ρ_S, a constant density; C_S, the specific heat; h, the depth of the layer; $\mathbf{v}_S$, the horizontal velocity of the ocean current; K_S, a constant diffusion coefficient; E_S, the energy added by radiation; G_2, the sensible heat given off to the atmosphere by vertical turbulent transport; G_3, heat lost by evaporation and W, the heat transported down through the bottom of the layer.

It is assumed that the initial surface ocean temperature has a departure from normal, but that the atmospheric conditions, the ocean currents and the temperature at the bottom of the thermocline are normal. This appears at first glance to be over simplified. However, from observational studies and from the numerical experiments carried out (Adem, 1969, 1970) it has become increasingly evident that the prediction of the anomalies of temperature for periods of a month or a season depends very strongly on the initial values of the anomalies themselves. One can, therefore, consider that the model which is studied in this paper is the simplest realistic model to start with; and is suitable for later use in building up a more general theory.

It is assumed that the normal[1] ocean temperature (denoted by T_{S_N}) satisfies equation (1) using normal values of the parameters and heating functions.

Subtracting from (1) the corresponding equation for T_{S_N}, using the above assumptions and the same linear heating functions as in the previous paper (Adem, 1970), one obtains the following equation for the anomalies of the surface ocean temperature:

$$\frac{\partial T_D}{\partial t} = AD + TU + HE \tag{2}$$

where $T_D = T_S - T_{S_N}$ is the anomaly of surface ocean temperature (or departure from normal), $AD = - \mathbf{v}_{S_T} \cdot \nabla T_D$ and $TU = K_S \nabla^2 T_D$.

[1] Normal values are the long-term means at each geographical point.

The term HE in (2) can be written as

$$HE = -(A + A' + A'')\, T_D \tag{3}$$

where AT_D is the heat lost by evaporation and by vertical turbulent transport of sensible heat at the surface, $A'T_D$ the heat lost at the bottom of the layer, and $A''T_D$ the heat lost by longwave radiation. The coefficients A, A' and A'' are given by

$$A = \frac{0.981\,(K_4 B + K_3)}{\rho_S\, C_S}\ \frac{V_{A_N}}{h}\ T_D \tag{4}$$

$$A' = -\frac{w_h}{h} \tag{5}$$

$$A'' = -\frac{1}{h}\ \frac{4\sigma T_{SO}^3}{\rho_S C_S} \tag{6}$$

where V_{A_N} is the normal surface air wind speed, w_h is the downward velocity at the bottom of the layer, σ the Stefan-Boltzmann constant, T_{SO} a constant mean surface temperature and K_4, B and K_3 are constants.

To evaluate A, A' and A'' the same numerical values as in the previous paper (Adem, 1970) will be used. Namely:

$C_S = 1$ cal gm^{-1}, $\rho_S = 1$ gm cm^{-3},

$w_h = -2 \times 10^{-5}$ cm sec^{-1}, $T_{SO} = 288$ K,

$\sigma = (8215) \times 10^{-14}$ cal cm^{-2} K^{-4} min^{-1},

$K_4 = 40.5 \times 10^{-3}$, $K_3 = 26.8$ gm sec^{-2} cm^{-1} K^{-1},

and $B = 1.28 \times 10^3$ gm sec^{-2} K^{-1}

The substitution of these numerical values in (4), (5) and (6) yields

$$A = 1.88 \times 10^{-6}\ V_{A_N} / h \tag{7}$$

$$A' = -2 \times 10^{-5} \ / \ h \tag{8}$$

$$A'' = -1.53 \times 10^{-6} \ / \ h \tag{9}$$

Comparison of the values of A, A' and A" as given by (7), (8) and (9) respectively, shows that A' and A" are negligibly small compared with A. Therefore, in this case one can write

$$HE = - A \ T_D \tag{10}$$

3. SERIES SOLUTION FOR Eq. (2).

Assuming that v_{ST}, K_S and A are constant parameters one can obtain a series solution for T_D, as follows:

Eq. (2) can be written for brevity:

$$\frac{\partial T_D}{\partial t} = L \, (T_D) \tag{11}$$

where L is an operator given by

$$L = - \mathbf{v}_{ST} \cdot \nabla + K_S \nabla^2 - A \tag{12}$$

Differentiating (11) with respect to time:

$$\frac{\partial T_D^2}{\partial t^2} = L \ \frac{\partial T_D}{\partial t} \tag{13}$$

Substituting (11) in (13) one obtains

$$\frac{\partial^2 T_D}{\partial t^2} = L \, (L \, (T_D)\,) \equiv L^2 \, (T_D)$$

Similarly,

$$\frac{\partial^3 T_D}{\partial t^3} = L^2 \left(\frac{\partial T_D}{\partial t} \right) = L^2 \left(L \left(T_D \right) \right) \equiv L^3 \left(T_D \right)$$

The n^{th} derivative is given by

$$\frac{\partial^n T_D}{\partial t^n} = L^{n-1} \left(L \left(T_D \right) \right) = L^n \left(T_D \right)$$

Given the initial values of the temperature anomalies T_{D_O}, the values at a later time T_D can be computed using a series expansion, as follows:

$$T_D = T_{D_O} + L \left(T_{D_O} \right) \Delta t + \tfrac{1}{2} L^2 \left(T_{D_O} \right) \Delta t^2 + \ldots + \frac{1}{n!} L^n \left(T_{D_O} \right) \Delta t^n + \ldots \ldots \tag{14}$$

The horizontal space field of the initial temperature anomaly can be represented by a double Fourier series:

$$T_{D_O} = \sum_{M=O}^{\infty} \sum_{N=O}^{\infty} T_{D_{NM_O}} \tag{15}$$

where, for $N > O$ and $M > O$,

$$\begin{aligned} T_{D_{NM_O}} = & \left(A_{NM_O} \sin \beta_M y + B_{NM_O} \cos \beta_M y \right) \sin \alpha_N x \\ & + \left(C_{NM_O} \sin \beta_M y + D_{NM_O} \cos \beta_M y \right) \cos \alpha_N x \end{aligned} \tag{16}$$

where

$$\alpha_N = \frac{N\pi}{x_O}$$

$$\beta_M = \frac{M\pi}{y_O}$$

and where $x = \pm x_O$ and $y = \pm y_O$ are the boundaries of the region of integration.

For the particular case M = 0, N > 0:

$$(T_{D_{NM_O}})_{M=O} = (B_{NM_O})_{M=O} \sin \alpha_N x + (D_{NM_O})_{M=O} \cos \alpha_N x$$

Similarly for N=O, M>O:

$$(T_{D_{NM}})_{N=O} = (C_{NM_O})_{N=O} \sin \beta_M y + (D_{NM_O})_{N=O} \cos \beta_M y$$

Finally, for N = O and M = O:

$$(T_{D_{NM_O}})_{N=O,\, M=O} = (D_{NM_O})_{N=O,\, M=O}\, .$$

The coefficients A_{NM_O}, B_{NM_O}, C_{NM_O} and D_{NM_O} are computed using a method described by Churchill (1941, p. 116) and for the sake of brevity will be omitted.

Substituting (15) in (14) one obtains

$$T_D = \sum_{M=O}^{\infty} \sum_{N=O}^{\infty} T_{D_{NM}} \tag{18}$$

where

$$T_{D_{NM}} = T_{D_{NM_O}} + L\,(T_{D_{NM_O}})\,\Delta t + \tfrac{1}{2}\, L^2\,(T_{D_{NM_O}})\,\Delta^2 t + \ldots + \frac{1}{n!}\, L^n (T_{D_{NM_O}})\,\Delta t^n + \ldots\ldots \tag{19}$$

Applying the operator (12) to (16), one obtains

$$L(T_{D_{NM_O}}) = (A_{NM_1} \sin \beta_M y + B_{NM_1} \cos \beta_M y) \sin \alpha_N x + (C_{NM_1} \sin \beta_M y + D_{NM_1} \cos \beta_M y) \cos \alpha_N x \tag{20}$$

in which

$$
\left.
\begin{aligned}
A_{NM_1} &= A_{NM_O}\gamma_{NM} + B_{NM_O} S_{V_M} + C_{NM_O} S_{U_N} \\
B_{NM_1} &= B_{NM_O}\gamma_{NM} - A_{NM_O} S_{V_M} + D_{NM_O} S_{U_N} \\
C_{NM_1} &= C_{NM_O}\gamma_{NM} - C_{NM_O} S_{V_M} - A_{NM_O} S_{U_N} \\
D_{NM_1} &= D_{NM_O}\gamma_{NM} - C_{NM_O} S_{V_M} - B_{NM_O} S_{U_N}
\end{aligned}
\right\} \quad (21)
$$

where

$$\gamma_{NM} = -A - K_S(\alpha_N^2 + \beta_M^2) \quad (22)$$

$$S_{V_M} = v_S \beta_M \quad (23)$$

$$S_{U_N} = u_S \alpha_N \quad (24)$$

and where u_S and v_S are the components of $\mathbf{v}_{S_T}$ in the x and y directions respectively.

In general one finds that

$$
\begin{aligned}
L^n (T_{D_{NM_O}}) = {} & (A_{NM_n} \sin\beta_M y + B_{NM_n} \cos\beta_M y) \sin\alpha_n x \\
& + (c_{NM_n} \sin\beta_M y + D_{NM_n} \cos\beta_M y) \cos\alpha_N x
\end{aligned} \quad (25)
$$

where

$$
\begin{aligned}
A_{NM_n} &= A_{NM_{n-1}}\gamma_{NM} + B_{NM_{n-1}} S_{V_M} + C_{NM_{n-1}} S_{U_N} \\
B_{NM_n} &= B_{NM_{n-1}}\gamma_{NM} - A_{NM_{n-1}} S_{V_M} + D_{NM_{n-1}} S_{U_N} \\
C_{NM_n} &= D_{NM_{n-1}}\gamma_{NM} + D_{NM_{n-1}} S_{V_M} - A_{NM_{n-1}} S_{U_N} \\
D_{NM_n} &= D_{NM_{n-1}}\gamma_{NM} - C_{NM_{n-1}} S_{V_M} - A_{NM_{n-1}} S_{U_N}
\end{aligned} \quad (26)
$$

Using (16), (20) and (25) the series expansion (19) can be written

$$T_{D_{NM}} = (A_{NM_O} + A_{NM_1}\Delta t + \tfrac{1}{2} A_{NM_2}(\Delta t)^2 + \ldots + \frac{1}{n!} A_{NM_n} (\Delta t)^n + \ldots)$$

$$\sin \beta_M y \sin \alpha_N x$$

$$+(B_{NM_O} + B_{NM_1}\Delta t + \tfrac{1}{2} B_{NM_2} (\Delta t)^2 + \ldots + \frac{1}{n!} B_{NM_n}(\Delta t)^n + \ldots)$$

$$\cos \beta_M y \sin \alpha_N x \qquad (27)$$

$$+ (C_{NM_O} + C_{NM_1}\Delta t + \tfrac{1}{2} C_{NM_2}(\Delta t)^2 + \ldots + \frac{1}{n!} C_{NM_n}(\Delta t)^n + \ldots)$$

$$\sin \beta_M y \cos \alpha_N x$$

$$+(D_{NM_O} + D_{NM_1}\Delta t + \tfrac{1}{2} D_{NM_2} (\Delta t)^2 + \ldots + \frac{1}{n!} D_{NM_n'}(\Delta t)^n + \ldots)$$

$$\cos \beta_M y \cos \alpha_N x$$

Formulas (21), (22), (23), (24), (25), (26) and (27) are valid when $M=0$, $N>0$ and when $N=0$, $M>0$ provided that one sets $A_{NM_O} = C_{NM_O} = \beta_M = 0$; and $B_{NM_O} = D_{NM_O} = \alpha_N = 0$ respectively.

Formula (26) can be used to compute the coefficients of the series expansion (27). For $n = 2$ one finds:

$$A_{NM_2} = A_{NM_O}(\gamma_{NM}^2 - S_{V_M}^2 - S_{U_N}^2) + 2 B_{NM_O} \gamma_{NM} S_{V_N}$$

$$+ 2 C_{NM_O} \gamma_{NM} S_{U_N} + D_{NM_O} S_{V_M} S_{U_N}$$

$$B_{NM_2} = B_{NM_O} (\gamma_{NM}^2 - S_{V_M}^2 - S_{U_N}^2) - 2 A_{NM_O} \gamma_{NM} S_{V_M} \qquad (28)$$

$$+ 2 D_{NM_O} \gamma_{NM} S_{U_N} - 2 C_{NM_O} S_{U_N} S_{V_M}$$

$$C_{NM_2} = C_{NM_O} (\gamma_{NM}^2 - S_{V_M}^2 - S_{U_N}^2) + 2D_{NM_O} \gamma_{NM} S_{V_M}$$

$$-2 A_{NM_O} \gamma_{NM} S_{U_N} - 2B_{NM_O} S_{U_N} S_{V_N}$$

$$D_{NM_2} = D_{NM_O} (\gamma_{NM}^2 - S_{V_M}^2 - S_{U_N}^2) - 2 C_{NM_O} \gamma_{NM} S_{V_M}$$

$$-2 B_{NM} \gamma_{NM} S_{U_N} + 2 A_{NM_O} S_{U_N} S_{V_N}$$

We shall define the following non-dimensional variables:

$$\tau_A = A\,\Delta t \tag{29}$$

$$\tau_{K_{NM}} = K_S\,(\alpha_N^2 + \beta_M^2)\,\Delta t \tag{30}$$

$$\tau_{U_N} = u_S \alpha_N\,\Delta t \tag{31}$$

$$\tau_{V_M} = v_S\,\beta_M\,\Delta t \tag{32}$$

Substituting (16), (21) and (28) in (27) and using (29), (30), (31) and (32) one obtains

$$\begin{aligned}
T_{D_{NM}} = {} & T_{D_{NM_O}} + T_{D_{NM_O}}\left(-\tau_A + \tfrac{1}{2}\tau_A^2 - \ldots\right) + \\
& T_{D_{NM_O}}\left(-\tau_{K_{NM}} + \tfrac{1}{2}\tau_{K_{NM}}^2 - \ldots\right) + \\
& \left(E_{NM_O}\,\tau_{U_N} - \tfrac{1}{2}T_{D_{NM_O}}\,\tau_{U_N}^2 + \ldots\right) + \\
& \left(F_{NM_O}\,\tau_{V_M} - \tfrac{1}{2}T_{D_{NM_O}}\,\tau_{V_M}^2 + \ldots\right) + \\
& T_{D_{NM_O}}\,\tau_A\,\tau_{K_{NM}} + E_{NM}^{*}\left(\tau_A + \tau_{K_{NM}}\right)\tau_{U_N} + \\
& F_{NM}\left(\tau_A + \tau_{K_{NM}}\right)\tau_{V_M} + G_{NM}\,\tau_{U_N}\,\tau_{V_M} + \ldots
\end{aligned}$$

where

$$\begin{aligned}
E_{NM_O} = {} & C_{NM_O}\sin\alpha_N x \sin\beta_M y + D_{NM_O}\sin\alpha_N x\cos\beta_M y \\
& - A_{NM_O}\cos\alpha_N x\sin\beta_M y - B_{NM_O}\cos\alpha_N x\cos\beta_M y \\
F_{NM_O} = {} & B_{NM_O}\sin\alpha_N x\sin\beta_M y - A_{NM_O}\sin\alpha_N x\cos\beta_M y \\
& + D_{NM_O}\cos\alpha_N x\sin\beta_M y - C_{NM_O}\cos\alpha_N x\cos\beta_M y \\
G_{NM_O} = {} & D_{NM_O}\sin\alpha_N x\sin\beta_M y - C_{NM_O}\sin\alpha_N x\cos\beta_M y \\
& - B_{NM_O}\cos\alpha_N x\sin\beta_M y + A_{NM_O}\cos\alpha_N x\cos\beta_M y
\end{aligned}$$

(33) is the solution of (2) corresponding to the initial anomaly T_{NM_O}. The solution corresponding to the arbitrary initial anomaly given by (15), is obtained by substituting (33) in (18).

Solution (33) depends on the four non-dimensional variables τ_A, τ_{KNM}, τ_{UN}, and τ_{VM}. When these variables are small enough so as to neglect the quadratic terms, the solution becomes

$$T_{D_{NM}} = T_{D_{NM_O}} (1 - \tau_A - \tau_{KNM}) + E_{NM_O} \tau_{UN} + F_{NM_O} \tau_{V_M} \quad (34)$$

which is equivalent to the solution obtained for the initial anomaly $T_{D_{NM_O}}$, using a time step equal to Δt with the Euler finite difference formula for the time derivative in (2).

When $u_S = v_S = 0$, (33) becomes

$$T_{D_{NM}} = T_{D_{NM_O}} (1 - \tau + ½ \tau^2 - ⅙ \tau^3 + \ldots + \frac{(-1)^n}{n!} \tau^n + \ldots) \quad (35)$$

where $\tau = \tau_A + \tau'_{KNM}$

Solution (35) can briefly be written

$$T_{D_{NM}} = T_{D_{NM_O}} e^{-\tau} \quad (36)$$

In this case formula (34) becomes:

$$T_{D_{NM}} = T_{D_{NM_O}} (1 - \tau) \quad (37)$$

A numerical study on the dependence of Δt on A, K_S, v_{S_T} and K_{NM} will be made. Therefore it is convenient to explore first the effect of each of these factors alone.

4. MODEL WITH HEATING ALONE.

When $A \neq 0$ and $K_S = v_S = u_S = 0$, Eq. (2) becomes

$$\frac{\partial T_D}{\partial t} = - A\, T_D \quad (38)$$

where A is given by (7).

In this case the solution is given by (35) with $\tau = \tau_A$

Substituting (7) in (29), one obtains

$$\tau_A = 1.88 \times 10^{-6} \, (V_{A_N} / h) \, \Delta t \tag{39}$$

$$\Delta t = (10^6 / 1.88) \, \tau_A \, (h / V_{A_N}) \tag{40}$$

In order that formula (37) could be used it is required that $e^{-\tau_A} - (1 - \tau_A) \ll 1$ and the choice of τ_A determines the truncation error. Therefore τ_A, can be regarded as a parameter and (40) represents a linear relation between Δt and h / V_{A_N}.

Table I gives the values of the exact solution $e^{-\tau_A}$, the approximate solution $(1 - \tau_A)$ and the truncation error $e^{-\tau_A} (1 - \tau_A)$ for several values of τ_A.

Using equation (40) and the values of Table I, one can evaluate the exact solution and the truncation error as a function of Δt. Fig. 1 shows the exact solution for the values of V_{A_N} shown in the corresponding curves. From this figure it follows that the isolated effect of heat lost by evaporation and sensible heat given off to the atmosphere is to reduce the anomaly, representing a tendency to return to normal. This is to be expected from the physical assumptions of normal conditions in the atmosphere and no horizontal transport of heat. Under these conditions the heat lost by evaporation and that given off to the atmosphere by vertical turbulent transport became proportional to the anomalies of temperature, and therefore reduce them.

It is of interest to point out that the observed maps of anomalies of ocean temperature do show this tendency for return to normal. As has been previously shown for a sample of 46 cases (Adem, 1969, 1970), one can use this tendency to make a prediction of the month-to-month change of the sign of the anomalies, which has skill over chance.

In previous works (Adem, 1964, 1965, 1970) a model of this type has been used which also takes into account the effect of the anomalies of the atmospheric temperature, which in the present

TABLE I. Values of T_D/T_{D_o} computed from the exact solution (second column) and the approximate one (third column), for different values τ_A (first column). The last column shows the difference of the two solutions (the truncation error when using (37)).

τ_A	$e^{-\tau_A}$	$1-\tau_A$	$e^{-\tau_A} - 1(1-\tau_A)$
3	.05	−2.00	2.05
2	.14	−1.00	1.14
3/2	.22	− .50	.72
1	.37	.00	.37
2/3	.47	.25	.22
1/2	.61	.50	.11
2/5	.67	.60	.07
1/3	.72	.67	.05
2/7	.75	.71	.04
1/4	.78	.75	0.3
1/5	.82	.80	.02
1/6	.84	.83	.01
1/7	.87	.86	.01
1/8	.89	.88	.01
1/9	.90	.89	.01
1/10	.90	.90	.00
1/16	.94	.94	.00
1/20	.95	.95	.00

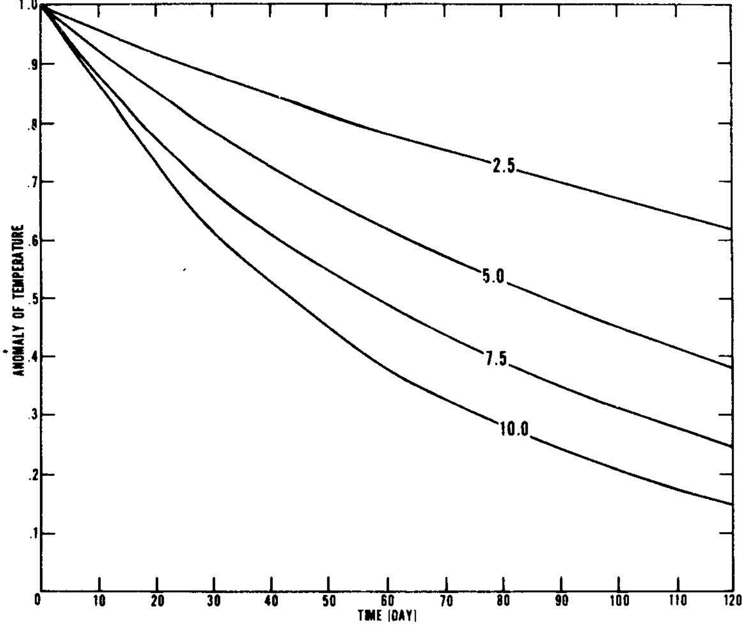

Figure 1. Prediction of the anomalies of surface ocean temperature based on the effect of the heat lost by evaporation and by vertical turbulent transport of sensible heat. Each curve is labeled with the corresponding value of the surface wind speed (m sec^{-1}). The ordinate values are the anomalies of temperature divided by their initial value at the same point.

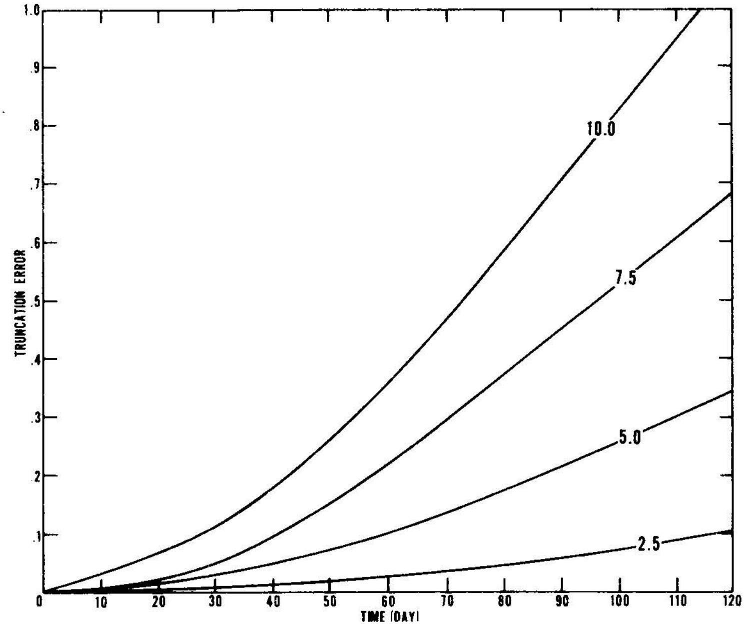

Figure 2. Truncation errors in the prediction of surface ocean temperature anomalies based on a model with heating alone when using the approximate formula (37) instead of the

discussion are taken as zero. Such a model yields a somewhat higher skill than simple return to normal.

Fig. 2 shows the truncation error when (37) is used instead of (35). Each curve is labeled with the wind speed in meters per second with h set equal to 100 meters.

So far, the assumption that h = 100 m has been used. However, since the solution depends on the ratio V_{AN}/h one can use Figs. 1 and 2 for values of h different from 100 m, provided that one changes the labels of the curves. For example, when h = 50 m the curves correspond to a value of V_{A_N} equal to half of those shown.

If one arbitrarily sets up the value $0.1 T_{D_O}$ as the maximum permissible truncation error, using the Euler finite-difference formula for the time derivative in (38), it follows from Fig. 2 that a single time step of a month can be used, provided that $V_{A_N} / h \leqq 0.1$ sec^{-1}. Therefore, one has the restriction that $V_{A_N} \leqq 10$ m sec^{-1} in a model in which h = 100 m; and that $V_{AN} \leqq 5$ m sec^{-1} in a model in which h = 50 m.

5. MODEL WITH HORIZONTAL MIXING ALONE.

In this case $K_S \neq 0$ and $A = v_S = u_S = 0$. Therefore equation (2) becomes:

$$\frac{\partial T_D}{\partial t} = K_S \nabla^2 T_D \tag{41}$$

The solution is given by (35) with $\tau = \tau_{K_{NM}}$

The non-dimensional variable $T_{K_{NM}}$ is given by (30).

The solution for this case is of the same type as for the model with heating alone. However, the essential difference is that the non-dimensional variable $\tau_{K_{NM}}$ depends on the horizontal size (or scale) of the anomalies, represented by the parameters α_N and β_M in (17).

From (17) and (30) one obtains

$$\tau_{K_{NM}} = \frac{K_S \pi^2}{R^2_{NM}} \Delta t \tag{42}$$

or

$$\Delta t = \frac{\tau_{KNM}}{K_S} \quad \frac{R^2_{NM}}{\pi^2} \tag{43}$$

where

$$R_{NM} = \frac{(x_O/N)(y_O/M)}{((x_O/N)^2 + (y_O/M)^2)^{1/2}} \tag{44}$$

R_{NM} is a measure of the scale or size of the T_{MN_O} component of the initial anomaly.

From (42) it follows that both the exact solution (36) and the truncation error made when one uses (37), increase with the square of the size of the anomalies and decrease with the value of the austausch coefficient.

The essential characteristics of the solution can be studied using one term of the series expansion. Therefore, as an illustrative example the case when in (16) $A_{NMO} \neq O$, $B_{NMO} = C_{NMO} = D_{NMO} = O$, will be considered. Therefore,

$$T_{D_{NM_O}}, = A_{NM_O} \sin \frac{M\pi}{y_O} y \sin \frac{N\pi}{x_O} x \tag{45}$$

The function (45) is shown in Figure 3, for N = 2 and M = 3. The profiles of sin $(N\pi/x_O)$ x and sin $(M\pi/y_O)$ y are shown on the x and y axes respectively. The anomalies are rectangles with sides equal to x_O/N and y_O/M and with the diagonal equal to $(x_O/N)(y_O/M)/R_{NM}$.

In this study anomalies of the size illustrated in Figure 4 will be considered. The value of the sides of the anomalies of size I is 8.17×10^7 cm on a polar stereographic projection, which is the grid-distance used in the previous computations (Adem, 1970). Anomalies of sizes II, III, IV, V, VI and VII are squares with sides equal to 2, 3, 4, 5, 6 and 7 times the grid-distance. These square anomalies correspond to M=N in formula (45).

In the numerical studies two values for K_S will be used: $(3/4) \times 10^8$ and 3×10^8 cm^2 sec^{-1}. These values are in agreement with determinations of the horizontal eddy difusion coefficient by different authors. Montgo-

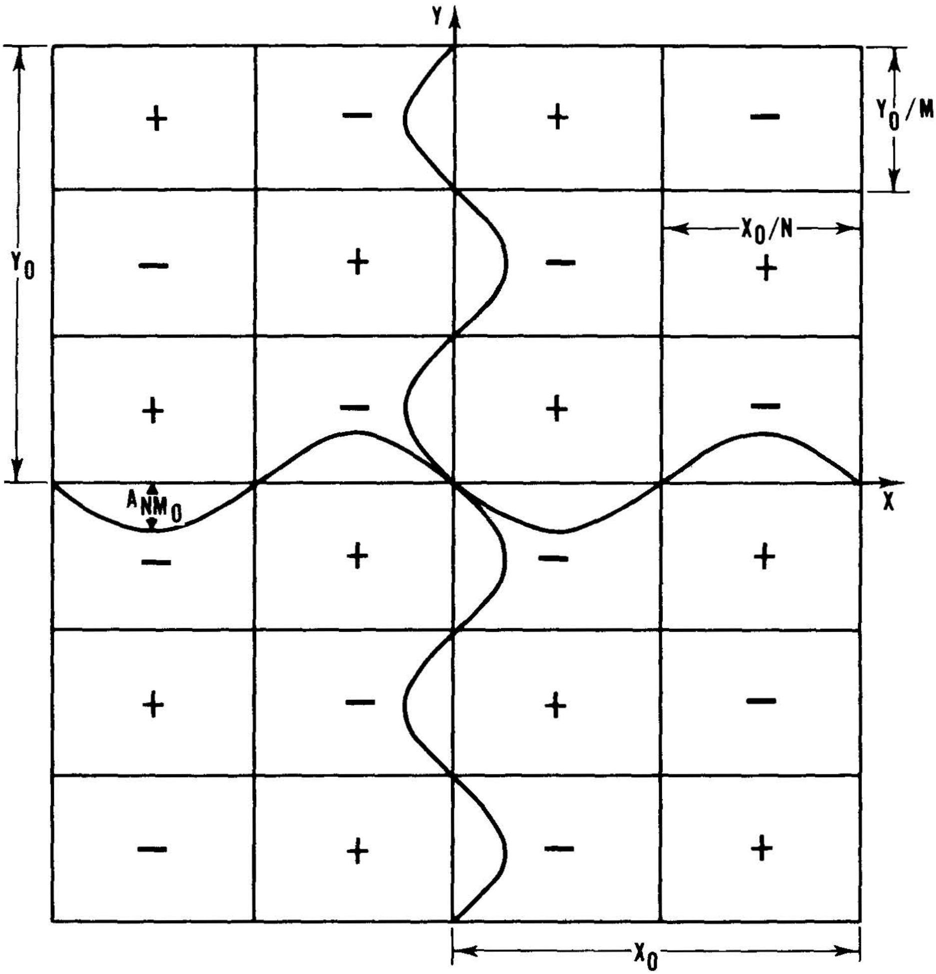

Figure 3. Anomaly distribution represented by the double-Fourier term $T_{D\,NM_o}$ as given by (45), for N = 2 and M = 3. The profiles of $A_{NM_o}\sin\,(N\pi/x_o)x$ and $A_{NM_o}\sin\,(M\pi/y_o)y$ are shown on the x and y axes respectively.

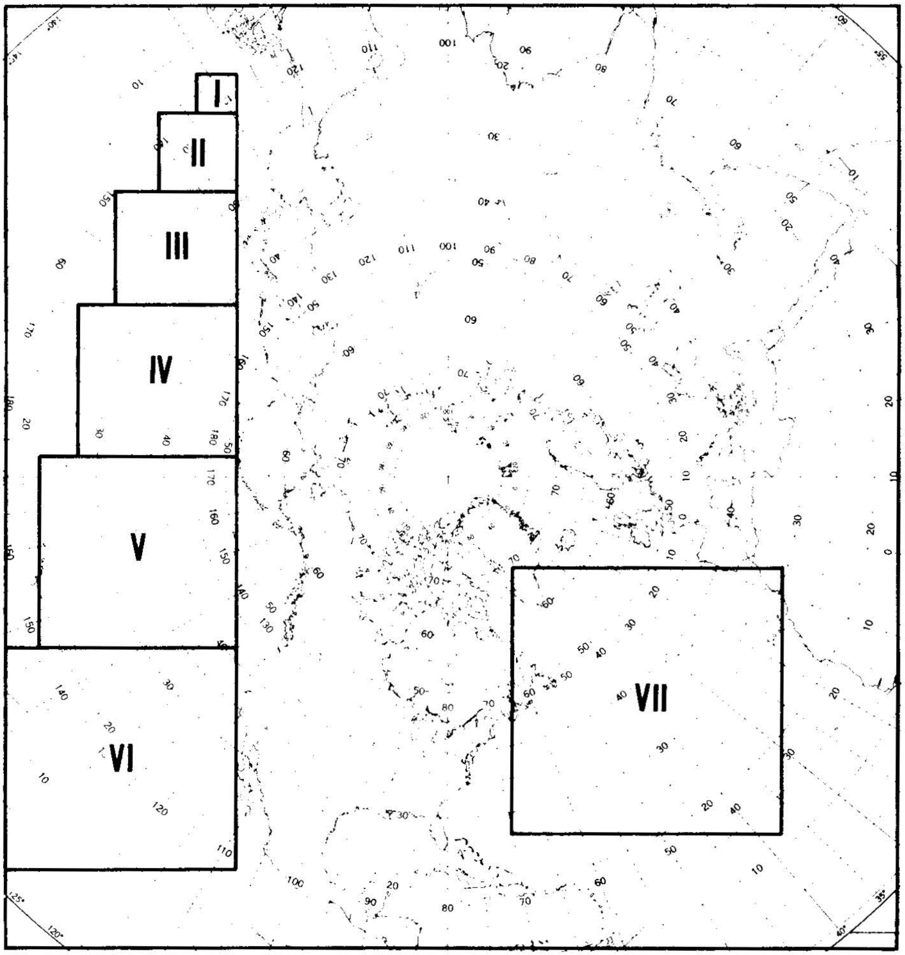

Figure 4. The seven sizes of anomalies of temperature considered, compared with the sizes of oceans and continents.

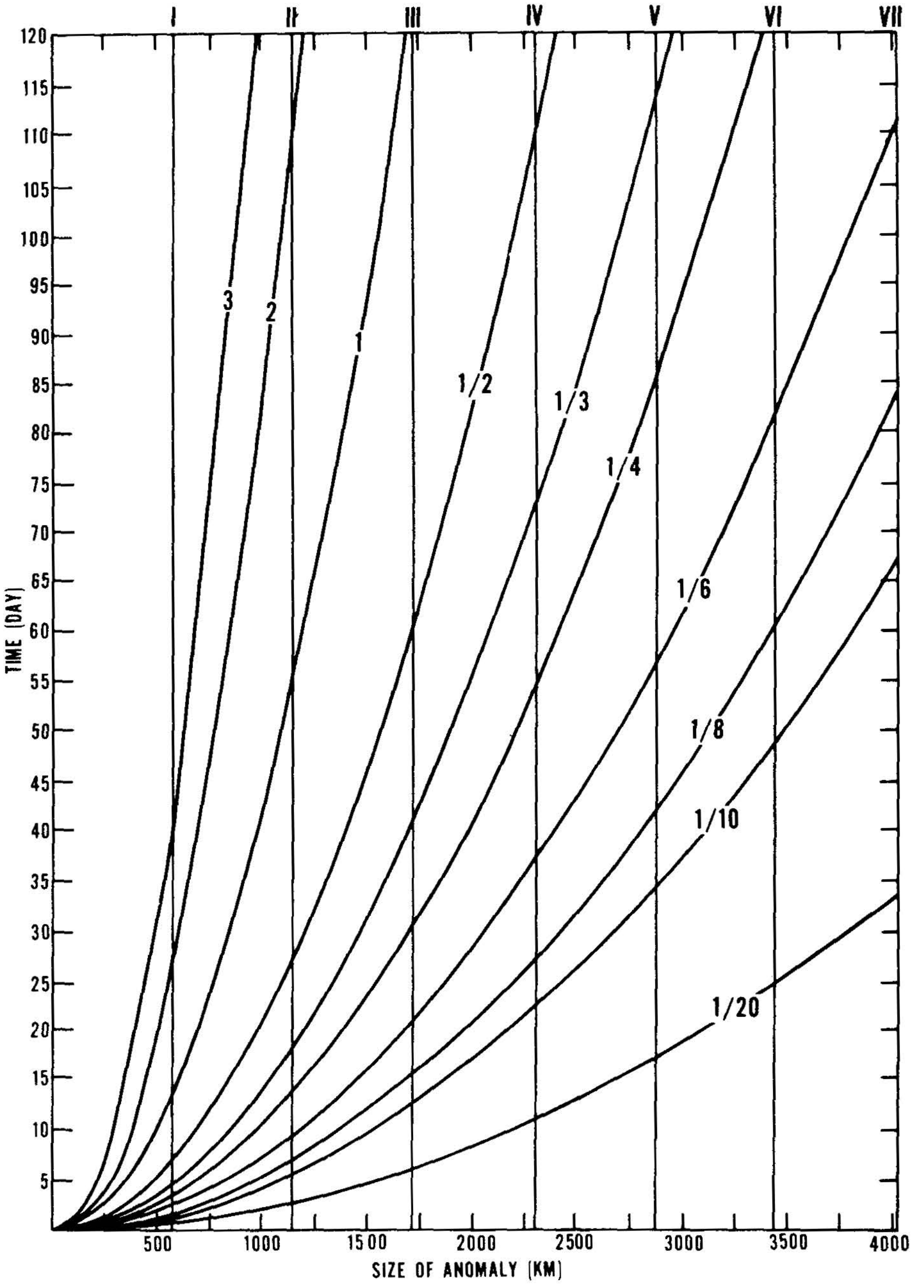

Figure 5. Family of curves of constant τ_{KNM} obtained from formula (43). The ordinate is the time (Δt) and the abscissa the size of the anomaly (R_{NM}). Each curve is labeled with the corresponding value of the non-dimensional variable τ_{KNM}. The vertical lines are those of constant values of R_{NM} corresponding to the sizes of anomalies shown in Figure 4; labeled with the corresponding Roman numeral.

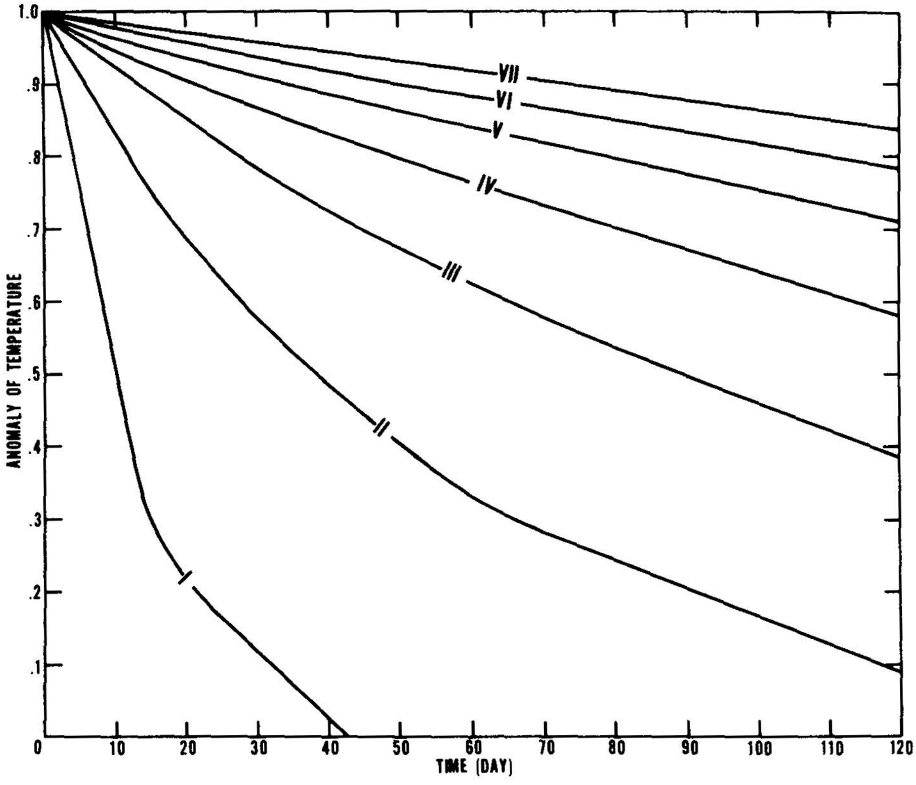

Figure 6. Prediction of the anomalies of surface ocean temperature based on the effect of horizontal mixing alone, for the different sizes of anomalies shown in Figure 4. Each curve is labeled with the corresponding Roman numeral. The ordinate values are the anomalies of temperature divided by their initial value at the same point.

mery (1939) obtained a value equal to 4×10^7 cm^{2}sec^{-1} in the Atlantic Equatorial Countercurrent in the surface layer of 200 meters depth. Neuman (1940) obtained a value equal to 5×10^8 cm^2 sec^{-1} in the surface waters east of the Grand Banks. Finally, Stommel (1950) found the lateral eddy diffusion coefficient in the Gulf Stream to be equal to 2.3×10^8 cm^2 sec^{-1}.

Fig. 5 shows the family of curves for $K_S = (3/4) \times 10^8$ cm^2 sec^{-1} representing the solution of Eq. (43) for different values of $T_{K\,NM}$ and R_{NM}. The ordinate is the time step (Δt) and the abscissa the size of the anomaly (R_{NM}). Each curve is labeled with the corresponding value of $\tau_{K\,NM}$. In the same figure are shown the vertical lines of constant values of R_{NM} corresponding to the sizes of anomalies illustrated in Fig. 4 and labeled with the corresponding Roman numeral.

The nomogram,(Fig. 5), together with Table I, can easily be used to obtain the exact solution (36) and the truncation error made when one uses (37). Since both (36) and (37) are proportional to $T_{D_{NM_O}}$ it is enough to consider a normalized solution $T_{D_{NM}} / T_{D_{NM_O}}$ in which the initial anomaly is equal to one.

Figure 6 shows the exact solution (36) for $K_S = (3/4) \times 10^8$ cm^2 sec^{-1} as a function of time, and for the 7 sizes of anomalies considered above. The curves are labeled with Roman numerals to designate the corresponding size of anomalies. According to these results the effect of turbulent mixing is to reduce the anomalies as the case of evaporation and vertical turbulent transport of sensible heat. However, the essential difference is that the rate of reduction of the anomalies is larger the smaller the size of the anomalies. Therefore, in the case of turbulent mixing, smaller anomalies disappear rapidly as time increases. For example, Fig. 6 shows that after 43 days anomalies of size I have disappeared. After 120 days anomalies of size II have been reduced to a negligible value of .09; while for the same time interval anomalies of size VII have been reduced only to a value of 0.89.

Therefore, in the case of turbulent mixing there is a smoothing of the initial temperature distribution in addition to a return to normal.

Fig. 7 shows the truncation errors for the case $K_S = (3/4) \times 10^8$ cm^2 sec^{-1} when one uses (37) instead of (36). According to this figure, if one accepts an error as large as 0.1 a prediction can be made with a single time step as large as 26 days, excluding anomalies of size smaller than II; up to 60 days, excluding sizes smaller than

III; and up to about 108 days excluding sizes smaller than IV.

From (43) it follows that if K_S is substituted by $C_1 K_S$ where C_1 is any arbitrary positive factor, then Figs. 5, 6 and 7 can be used provided one changes the time Δt to $\Delta t/C_1$ Therefore, when K_S is increased, by a factor of 4, to 3×10^8 $cm^2 sec^{-1}$, one has to replace Δt by $\Delta t/4$, so that the effect of turbulence is four times faster and the permissible time steps are four times smaller.

6. MODEL WITH TURBULENT HORIZONTAL MIXING AND HEATING

When $A \neq 0$, $K_S \neq 0$ and $v_{S_T} = 0$, Eq. (2) becomes

$$\frac{\partial T_D}{\partial t} = K_S \Delta^2 T_D + A T_D \tag{46}$$

In this case the solution is given by (36).

For given values of V_{A_N} and K_S, this case can be treated in a similar way as the model with horizontal mixing alone.

For the case when V_{A_N}=5 m sec^{-1} and $K_S = (\frac{3}{4}) \times 10^8$ cm^2 sec^{-1}, the values of the predicted anomalies using the exact formula (36) are shown in Fig. 8 for each size of anomaly. In the same figure the dashed line is the solution when there is no mixing (K_S = O).

Comparison of the curves of Fig. 8 for each size of anomaly with the dashed line and with the corresponding curves of Fig. 6 shows that the combined effect of heating and horizontal mixing is to accelerate the reduction of the anomalies.

Fig. 9 shows the truncation errors, for the same case of Fig. 8, when one uses the approximate formula (37). The dashed curve corresponds to the case when there is no horizontal mixing. Comparison of the numbered curves with the dashed curve and with the corresponding curves of Fig. 7 shows that the inclusion of both heating and horizontal mixing causes the truncation errors to be larger than those when only heating or horizontal mixing is present.

From Fig. 9 and again assuming that a truncation error of 0.1 is tolerable, it follows that a prediction can be made with a simple time step up to 18 days if anomalies of size smaller than II are excluded; up to 30 days excluding sizes smaller than III; and up to 50 days, excluding sizes smaller than VII.

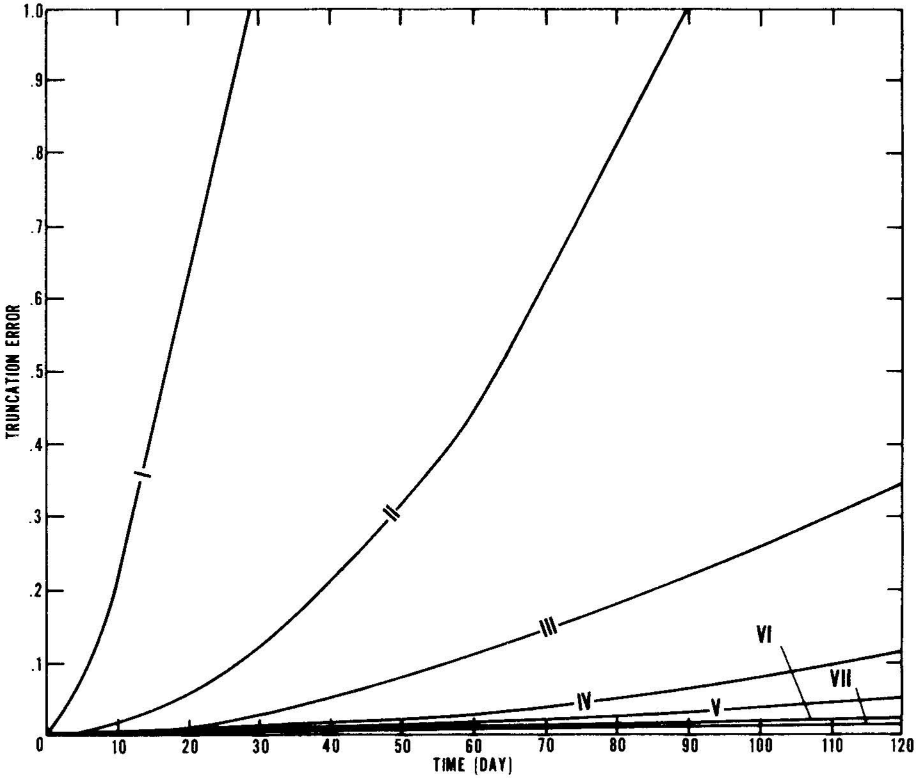

Figure 7. Truncation errors in the prediction of surface ocean temperature anomalies based on a model with horizontal mixing alone when using the approximate formula (37) instead of the exact formula (36) for the different sizes of anomalies shown in Figure 4. Each curve is labeled with the corresponding Roman numeral. The ordinate values are the truncation errors of the anomalies of temperature divided by the initial value of the anomalies at the same point.

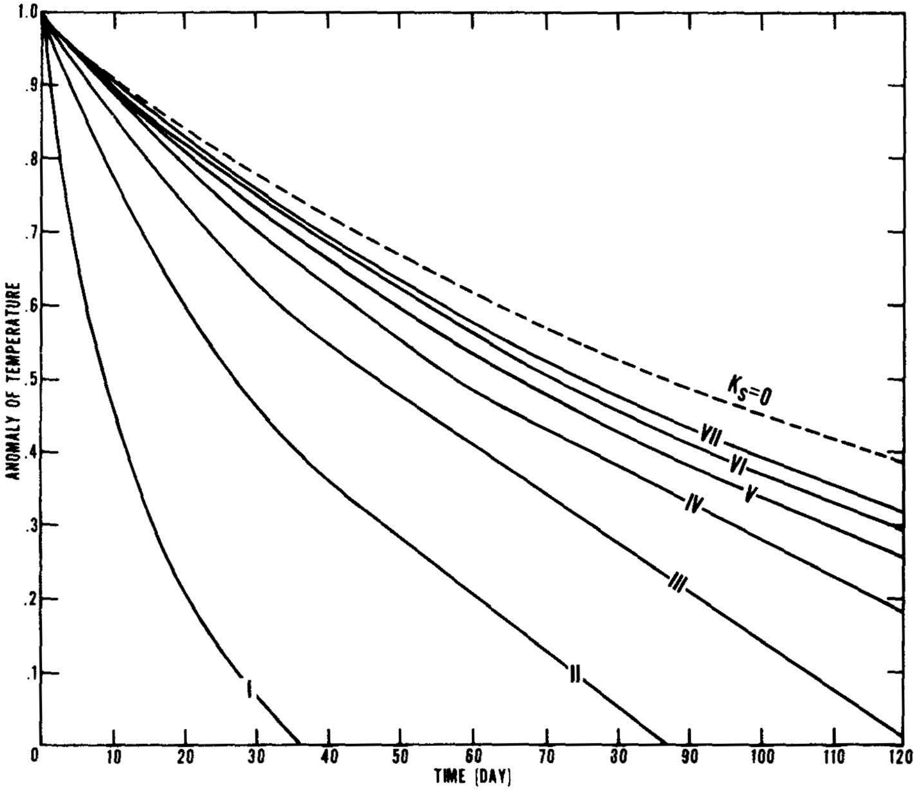

Figure 8. Prediction of the anomalies of surface ocean temperature based on the effect of horizontal mixing alone, for the different sizes of anomalies shown by Roman numeral in Figure 4. Each curve is labeled with the corresponding Roman numeral. The ordinate values are the anomalies of temperature divided by their initial value at the same point. The dashed curve represents the prediction for zero mixing coefficient.

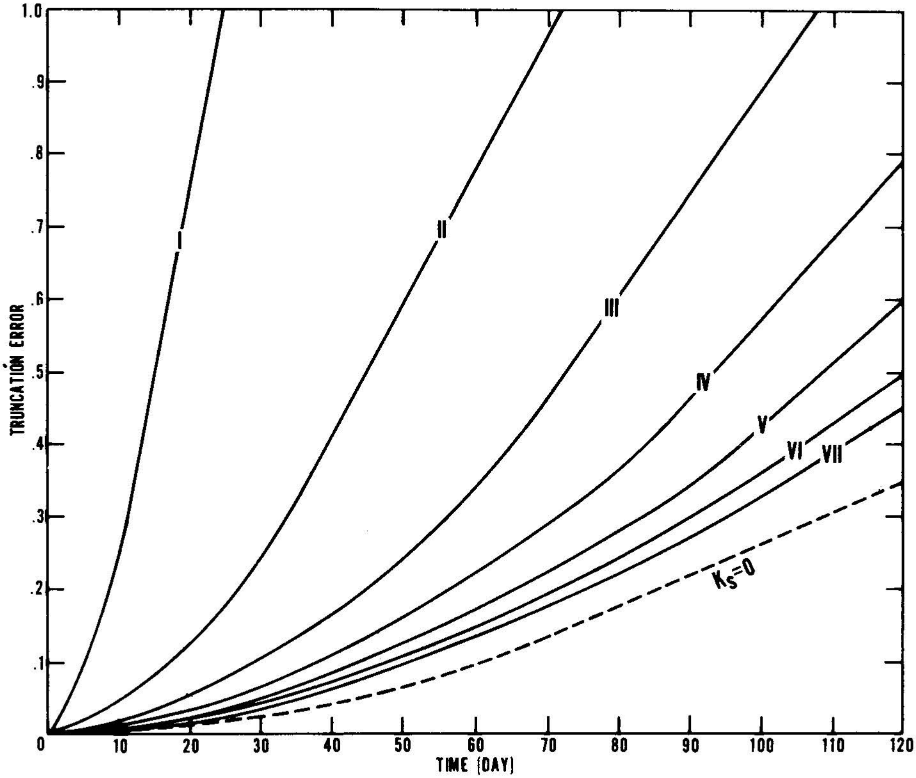

Figure 9. Truncation errors when using formula (37) instead of (36) in the prediction which includes only horizontal mixing and heating, for the different sizes of anomalies shown by Roman numeral in Fig. 4. Each curve is labeled with the corresponding numeral. The ordinate values are the truncation errors of the anomalies of temperature divided by the initial value of the anomalies at the same point. The dashed curve corresponds to zero mixing coefficient.

7. MODEL WITH TRANSPORT BY MEAN OCEAN CURRENTS ALONE.

It is of interest to study the isolated effect of the mean ocean currents.

In this case $A=K_S=O$ and equation (2) becomes

$$\frac{\partial T_D}{\partial t} = -u_S \frac{\partial T_D}{\partial x} - v_S \frac{\partial T_D}{\partial y} \tag{47}$$

where u_S and v_S are the components of $\mathbf{v}_{S_T}$ in the x and y direction respectively.

The solution of (47) is of the type

$$T_D = F(x-u_S t\,,\ y - v_S t) \tag{48}$$

Therefore, the anomaly of temperature which initially was at the point x, y and had the initial value F(x,y) at time $t = \Delta t$ has moved a distance $u_S\Delta t$ in the x direction and $v_S\Delta t$ in the y direction, without any change in its magnitude.

Assuming that the initial anomaly distribution is given by the double Fourier Series (15) and the components of the series by (16); and applying (48) to (16), the following solution is obtained:

$$T_{D_{NM}} = [A_{NM_O} \sin(\beta_M y - \tau_{V_M}) + B_{NM_O} \cos(\beta_M y - \tau_{V_M})] \sin(\alpha_N x - \tau_{U_N})$$
$$+ [C_{NM_O} \sin(\beta_M y - \tau_{V_M}) + D_{NM_O} \cos(\beta_M y - \tau_{V_M})] \cos(\alpha_N x - \tau_{U_N}) \tag{49}$$

where

$$\tau_{U_M} = u_S \alpha_N \Delta t \quad \text{and} \quad \tau_{V_M} = v_S \alpha_N \Delta t$$

When τ_{V_M} and τ_{U_N} are small enough, (49) becomes:

$$
\begin{aligned}
T_{D_{NM}} &= T_{D_{NM_O}} \left(1 - \tfrac{1}{2}\tau^2_{U_N} - \tfrac{1}{2}\tau^2_{V_M}\right) + \\
& E_{NM_O}\,\tau_{U_N} + F_{NM_O}\,\tau_{V_M} + G_{NM_O}\,\tau_U\,\tau_V
\end{aligned}
\tag{50}
$$

where only terms up to the quadratic ones have been included in the series expansions of sin τ_{V_M}, sin τ_{U_N}, cos τ_{U_N} and cos τ_{V_M}

Solution (50) is the same as (33) when $A=K_S = 0$ (or $\tau_A = T_{K_{NM_O}} = 0$).

Solutions (49) and (50) depend on the non-dimensional variables τ_{V_M} and τ_{U_N} which are given by

$$\tau_{V_M} = v_S \pi (y_O/M)^{-1} \Delta t \tag{51}$$

$$\tau_{U_N} = u_S \pi (x_O/M)^{-1} \Delta t \tag{52}$$

According to (51) the non-dimensional variable τ_{V_M} depends linearly on the y component of the ocean current (v_S) and is inversely proportional to y_O/M; while τ_{U_N} depends linearly on the x component of the current (u_S) and is inversely proportional to x_O/N.

Since (51) and (52) are similar, one needs to study only one of them. Therefore, it is enough to deal with (52), which can be written

$$\Delta t = (\tau_{U_N}/u_S)\,(x_O/N) \tag{53}$$

Fig. 10 shows a graphical representation of (53). The ordinate is the time step and the abscissa, x_O/N. Each line corresponds to a given value of the ratio τ_{U_N}/u_S, The lines are labeled with the value of τ_{U_N} corresponding to the case when $u_S = 0.1$ knot. The graph can be applied for any value of the velocity $u_S = u_1\, 10^{-1}$ knots, provided that one multiplies each of the values of τ_{U_N} by u_1.

For rectangular anomalies illustrated in Figure 5 (where $(x_O/N) = (y_O/M)$, the relation of x_O/N and the size of the anomalies, R_{NM} is given by

$$(x_O/N) = (R_{NM} / 2^{1/2}) \tag{54}$$

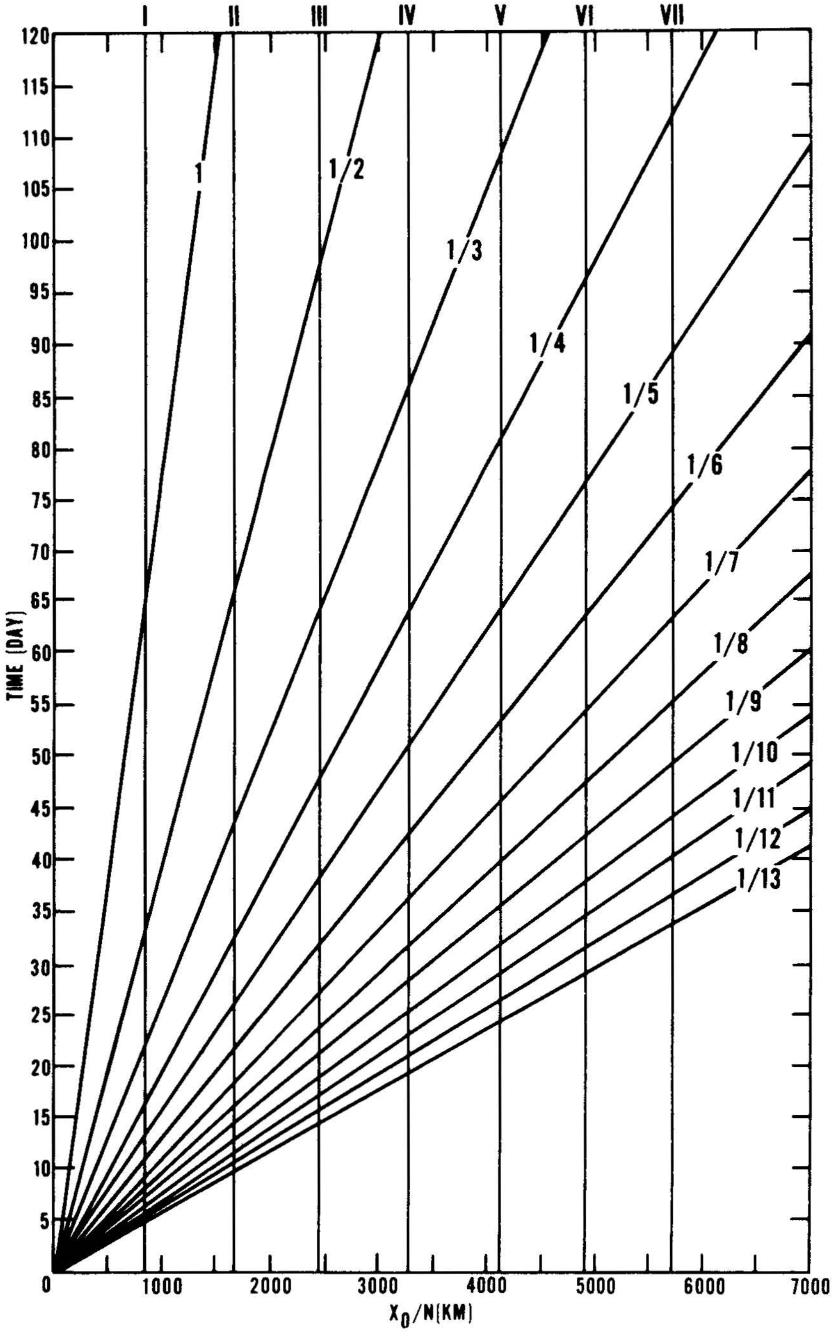

Figure 10. Family of lines of constant value of τ_{U_N} obtained from formula (53) for U_s=0.1 knots. The ordinate is the time (Δt) and the abscissa the distance (X_o/N). Each curve is labeled with the corresponding value of the non-dimensional variable τ_{U_N}. The vertical lines are those of constant value $(x_o/N)=(R_{NM}/2^{1/2})$ corresponding to the sizes of anomalies shown by Roman numerals in Figure 4 and labeled with the corresponding numeral.

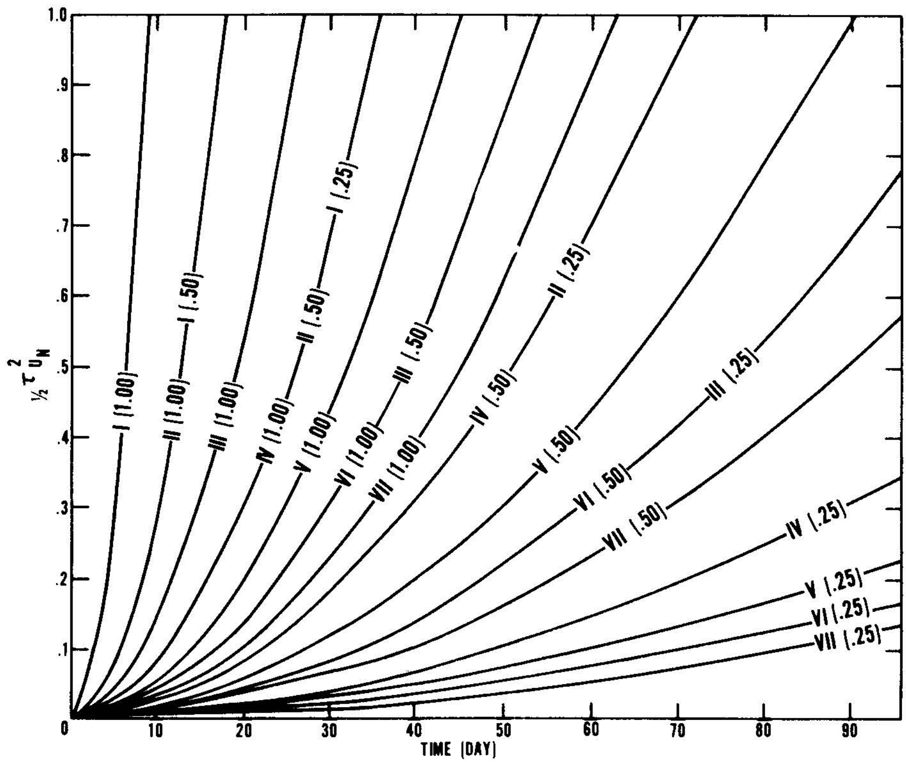

Figure 11. Maximum values of the truncation error (½) τ^2_{UN} in a prediction of surface ocean temperature anomalies in which formula (56) is used instead of (55), for the different sizes of anomalies and for ocean currents with speeds equal to 0.25, 0.50 and 1.00 knots.

The vertical lines corresponding to the values of x_O/N for the seven sizes of anomalies considered previously are shown in Figure 10, and labeled with the corresponding Roman numeral.

As in the previous cases, the assumption that $A_{NM_O} \neq O$ and $B_{NM_O} = C_{NM_O} = D_{NM_O} = O$ will be made. Furthermore, the case $u_S \neq O$ and $v_S = O$ will be considered. Therefore, solution (50) becomes:

$$T_{DNM} = A_{NM_O} \sin \beta_M y \left(\sin \alpha_N x (1 - \tfrac{1}{2} \tau^2_{U_N}) - \tau_{U_N} \cos \alpha_N x\right) \quad (55)$$

and for sufficiently small values of τ_{U_N}:

$$T_{DNM_O} = A_{NM_O} \sin \beta_M y \left(\sin \alpha_N x - \tau_{UN} \cos \alpha_N x\right) \quad (56)$$

which is equivalent to the use of Euler finite differences in equation (47).

Fig. (10) can be used to evaluate τ_{U_N} and (½) $\tau^2{}_{U_N}$ as functions of Δt for all sizes of anomalies and values of u_S.

Fig. (11) shows (½) τ^2_{UN} . The curves are labeled with Roman numerals to designate the corresponding size of anomalies, with a number in parenthesis to designate the value of u_S, Some curves are labeled 2 or 3 times because curves coincide for different combinations of anomaly size and current speed.

Formula (56) can be used provided (½) τ^2_{UN} is sufficiently small. As previously, the largest permissible truncation error is chosen to be equal to one tenth of the initial anomaly. Furthermore, since the higher powers of τ_{U_N} are negligible, (½) τ^2_{UN} is an excellent approximation of the truncation error.

According to Fig. 11, when the mean ocean current is equal to 0.5 knots, a prediction with formula (56) can be made up to about 11 days, excluding anomalies of sizes smaller than II; up to 16 days excluding sizes smaller than III; and up to 22 days, excluding sizes smaller than IV.

When the ocean current is equal to 0.25 knots the time steps increase to 22, 32 and 47 days, excluding respectively sizes smaller than II, III and IV; and when the ocean current is equal to 1.0 knot the time step decreases to 6, 8 and 10 days, excluding sizes smaller than II, III and IV, respectively.

The above results can be generalized to include the case of a current whose components are given by

$$u_S = V \cos\theta$$
$$v_S = V \sin\theta \tag{57}$$

where V is the speed of the current and θ the angle of its direction with the x– axis:

Substituting the values $A_{N M_O} \neq O$ $B_{N M_O} = C_{N M_O} = D_{N M_O} = O$ in (50) one finds that in this case the quadratic terms are given by

$$-\tfrac{1}{2}(\tau^2_{UN} + \tau^2_{VM}) + \tau_{UN}\,\tau_{VM} = -\tfrac{1}{2}V^2\left[\cos^2\theta\left(\frac{\pi\,\Delta t}{x_O/N}\right)^2 + \sin^2\theta\left(\frac{\pi\,\Delta t}{y_O/M}\right)^2\right] + \frac{V^2}{2}\sin 2\theta\,\frac{\pi^2(\Delta t)^2}{(x_O/N)(y_O/M)} \tag{58}$$

For the case when $(x_O/N) = (y_O/M)$ the maximum value of the right-hand side of (58) is obtained for $\theta = \pi/4$ and is equal to

$$-(\tfrac{1}{2})\,V^2\,[\pi\,\Delta t\,/\,(x_O/N)]^2$$

which is the same truncation error as $(\tfrac{1}{2})\ \tau^2_{UN}$ provided that one replaces u_S by V

A further generalization is possible by considering the cases when $(x_O/N) \neq (y_O/M)$. From (58) it follows that in this case the truncation error is smaller than when $(x_O/N) = (y_O/M)$ provided that one chooses as side of the squares, the smallest values of x_O/N and y_O/M.

The above analysis of the truncation error includes all the components of the series expansion (18) for ocean currents in any direction; provided one includes an adequate range of values of speed of currents and sizes of anomalies.

8. MODEL WITH ALL THE TERMS INCLUDED.

When A, K_S and v_{ST} are all different from zero, and the initial temperature field is given by the double Fourier series (15), the solution of equation (2) is given by the series (18) whose components are given by (33).

Solution (33) contains as particular cases the solution (36) for the case with only heating and horizontal mixing; and (50) for the case with only transport by mean ocean currents. For sufficiently small values of Δt, (33) can be written

$$T_{D_{NM}} = T_{D_{NM_O}}(x, y)\, e^{-(\tau_A + \tau_{K_{NM}})} + T_{D_{NM_O}}(x - u_S \Delta t, y - v_S \Delta t) - T_{D_{NM_O}}(x, y) \tag{59}$$

The first term in the right hand side of (59) corresponds to the combined effect of heating and horizontal mixing, discussed in Section 6; the last two terms represent the change in the anomaly due to the ocean currents, which was discussed in Section 7.

Retaining all terms up to the quadratic terms, solution (33) can be written:

$$T_{D_{NM}} = (T_{D_{NM}})_{AK} + (T_{D_{NM}})_{V_{S_T}} + E_{NM_O}(\tau_A + \tau_{K_{NM}})\, \tau_{U_N} + F_{NM_O}(\tau_A + \tau_{K_{NM}})\tau_{V_N} - T_{D_{NM_O}} \tag{60}$$

where $(T_{D_{NM}})_{AK}$ is the solution (36) in which $u_S = v_S = 0$ and $(T_{D_{NM}})_{V_{S_T}}$ is the solution (50) in which $A = K_S = 0$

The complete solution has therefore the additional interaction terms

$$(\tau_A + \tau_{K_{NM}})(E_{NM_O}\, \tau_{U_N} + F_{NM}\, \tau_{V_M}) \tag{61}$$

that contribute to the truncation error.

Assuming as before that $A_{NM_O} \neq O$, and $B_{NM_O} = C_{NM_O} = D_{NM_O} = O$, and that the ocean current is in the x direction, (33) becomes

$$T_{D_{NM}} = A_{NM_O} \sin \beta_M y \left\{ \sin \alpha_N x \left[1 - \tau_A - \tau_{K_{NM}} + (½)(\tau_A + \tau_{K_{NM}})^2 - (½)\tau^2_{U_N} \right] - \cos \alpha_N x \left[\tau_{U_N} (1 + \tau_A + \tau_{K_{NM}}) \right] \right\} \tag{62}$$

and the interaction term (61) is

$$-A_{NM} \ \tau_{UN} \ (\tau_A + \tau_{K_{NM}}) \cos \alpha_N x \ \sin \beta_M y \qquad (63)$$

The solution when all non-linear terms are neglected is

$$T_{D_{NM}} = A_{NM_O} \ \sin \beta_M y \ [(1 - \tau_A - \tau_{K_{NM}}) \sin \alpha_N x - \tau_{UN} \ \cos \alpha_N x] \qquad (64)$$

When (64) is used instead of (62) the truncation error is given by

$$\epsilon = A_{NM_O} \sin \beta_M y \ (\epsilon_1 \sin \alpha_N x - \epsilon_2 \ \cos \alpha_N x) \ (\Delta t)^2 \qquad (65)$$

where

$$\epsilon_1 = (½) \ [A + K_S \ (\alpha_N^2 + \beta_M^2)]^2 - (½) \ u_S^2 \ \alpha_N^2 \qquad (66)$$

$$\epsilon_2 = [A + K_S \ (\alpha_N^2 + \beta_M^2)] \ u_S \ \alpha_N \qquad (67)$$

Let $\epsilon = \epsilon^*$ be the maximum premissible truncation error, then

$$| \epsilon^* / A_{NM_O} | \leftharpoons | \sin \beta_M y \ | \ | \epsilon_1 \ \sin \alpha_N x - \epsilon_2 \ \cos \alpha_N x \ | \ (\Delta t)^2 \qquad (68)$$

To satisfy (68) for all values of x and y, one must choose $(\Delta t)^2$ corresponding to the case when $|\epsilon_1 \ \sin \alpha_N x - \epsilon_2 \cos \alpha_N x \ |$ and $| \sin \beta_M y |$ are maxima. Therefore,

$$\Delta t = \left(\frac{| \epsilon^* / A_{NM} \ |}{| \ \epsilon_1 \ \sin (\alpha_N x)_m - \epsilon_2 \ \cos (\alpha_N x)_m \ |} \right)^{½} \qquad (69)$$

where

$$(\alpha_N x)_m = \text{ang} \tan (- \epsilon_1 / \epsilon_2)$$

Computations of the permissible time step corresponding to a truncation error $|\epsilon^* / A_{NM}| = 0.1$ for different values of the parameters V_{A_N}, v_{S_T} and K_S and for the four smallest sizes of anomalies (I, II, III and IV) are shown in Table II.

The cases when only HE, TU, HE+TU or AD is included in Eq. (2), have been extensively studied previously, but are shown for comparative purposes. The results shown in Table II give values of the permissible time step about 10% smaller than those obtained in Sections 4, 5 and 6, because these results are based on the truncation errors due only to the quadratic terms.

The results when all the terms are included (HE+TU+AD) are summarized below:

1) The permissible time step is smaller than either of the cases when HE+TU or AD are included alone. This is due to he truncation errors of the interaction term containing the product τ_{U_N} (τ_A + $\tau_{K_{NM}}$).

2) It depends strongly on the smaller sizes of anomalies included in the prediction. For example, when $|v_{S_T}| = 0.5$ knots, $V_{A_N} = 10$ m sec^{-1} and $K_S = (3/4) \times 10^8$ cm^2 sec^{-1}, Δt is equal to 3.5 days when the lowest size is I and 7.8 days when it is II.

3) It depends weakly on V_{A_N}. For example, for size of anomaly II, $|v_{S_T}| = 0.5$ knots and $K_S = (3/4) \times 10^8$ cm sec^{-1}, Δt is equal to 8.8 days when $V_{A_N} = 5$m sec^{-1} and equal to 7.8 days when $V_{A_N} = 10$m sec^{-1}.

4) It depends strongly on the diffusion coefficient used (K_S). For example, for anomalies of size II, $|v_{S_T}| = 0.5$ knots and $V_{A_N} = 10$ m sec^{-1}, $\Delta \tau$ is equal to 7.8 days for $K_S = (3/4)\ 10^8$cm^2 sec^{-1} ; and equal to 4.3 days for $K_S = 3 \times 10^8$ cm^2 sec^{-1}.

5) It depends strongly on the speed of ocean currents. For example, for $K_S = (3/4) \times 10^8$ cm^2 sec^{-1} and $V_{A_N} = 10$m sec^{-1}. Δt is to equal to 7.8 days for $|v_{S_T}| = 0.5$ knots and 4.9 days for $|v_{S_T}| = 1.0$ knots.

Fig. 12 shows the Winter surface ocean current speed in knots for the North Atlantic Ocean as given by the U. S. Naval Oceanographic Office (1965). From this figure it is evident that over most of the oceans the assumption of using , $|v_{S_T}| = 0.5$ knots is a realistic one.

TABLE II. Time steps corresponding to a truncation error equal to one tenth of the maximum value of the initial anomaly, for anomalies of sizes I, II, III and IV.

Terms included	Value of parameters used			Time step (days)			
	V_{A_N} (m sec^{-1})	v_{S_T} (knots)	K_S ($10^8 cm^2 sec^{-1}$)	I	II	III	IV
HE	5	0	0	55.0	55.0	55.0	55.0
	10	0	0	27.5	27.5	27.5	27.5
TU	0	0	3/4	5.8	23.2	52.2	92.8
	0	0	3	1.4	5.8	13.0	23.2
HE+TU	5	0	3/4	5.3	16.3	26.8	34.6
	10	0	3/4	4.8	12.5	17.9	21.3
	5	0	3	1.4	5.3	11.2	16.3
	10	0	3	1.4	4.6	8.9	13.6
AD	0	0.5	0	5.3	10.6	15.9	21.2
	0	1.0	0	2.6	5.3	7.9	10.6
HE+TU+ AD	5	0.5	3/4	3.7	8.8	13.5	17.9
	10	0.5	3/4	3.5	7.8	11.9	14.9
	5	1.0	3/4	2.4	5.0	7.6	9.9
	10	1.0	3/4	2.3	4.9	7.2	9.4
	5	0.5	3	1.4	4.7	8.7	12.9
	10	0.5	3	1.3	4.3	7.7	10.8
	5	1.0	3	1.2	3.7	6.3	8.9
	10	1.0	3	1.2	3.5	5.9	8.1

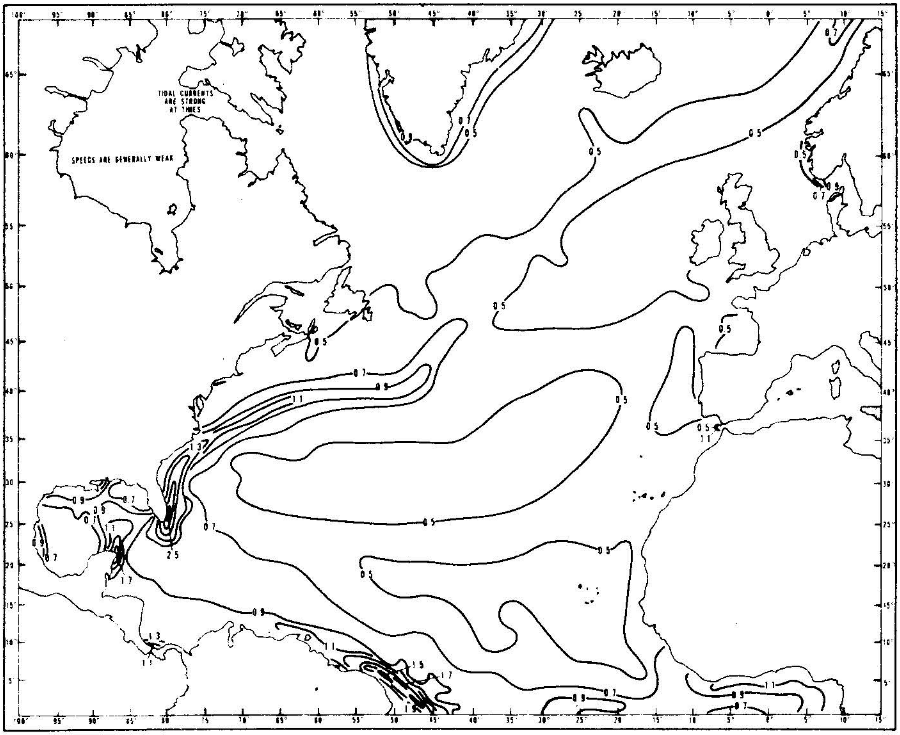

Figure 12. Mean current speed for Winter, in knots (from charts of the U. S. Naval Oceanographic Office).

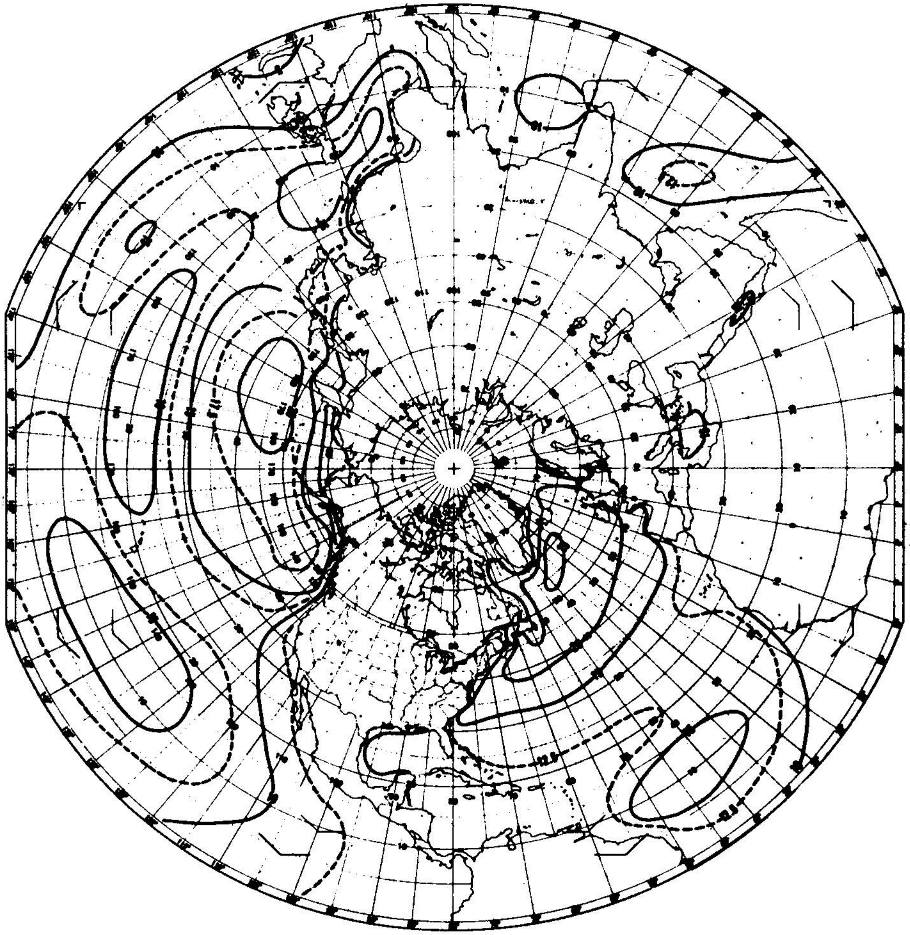

Figure 13. Surface wind speed, in knots (from data of McDonald)

Fig. 16 shows the Winter surface wind speed in knots as prepared by McDonald (1938). Since $V_{A_N}=10$ m sec^{-1} is equivalent to 19.4. knots, it is evident that this value of V_{A_N} is an upper limit for the values of the surface wind speed over most of the oceanic areas.

Therefore, if one excludes anomalies of size smaller than II and uses $|v_{S_T}| = 0.5$ knots and $V_{A_N} = 10$m sec^{-1} a time step equal to 7.8 days is acceptable over most of the oceans when $K_S = \frac{3}{4} \times 10^8$ cm^2 sec^{-1}. and equal to 4.3 days when $K_S = 3 \times 10^8$ cm^2 sec^{-1}.

Given a time step Δt_1, the corresponding truncation error is 0.1 $(\Delta t_1/\Delta t)^2$ where Δt is given by the values in Table II. Therefore, with the time step of 7.8 days when $K_S^4 = (\frac{3}{4}) \times 10^8$ $cm^2 sec^{-1}$, if one includes anomalies of size I their truncation error is equal to 0.5.

Furthermore, if instead of a time step of 7.8 days one uses a time step of 15 days the truncation errors are equal to .37, .16 and .10 for sizes of anomalies II, III and IV and smaller than 0.1 for anomalies larger than IV. In this case the anomalies of size I have to be excluded because the truncation error is 1.84 times the size of the initial anomalies.

9. FINAL REMARKS AND CONCLUSIONS

In previous papers (Adem, 1964, 1965, 1969) a time step of 30 days has been used. According to the present study this is justified in a model without horizontal transport by ocean currents and horizontal mixing.

In a recent paper (Adem, 1970) horizontal mixing and transport by mean ocean currents were included, using a 30 day time step. According to the present study, the resulting predictions will have truncation errors for the smaller sizes of anomalies which are larger than one tenth of the initial anomaly. Furthermore, the largest, and perhaps unacceptable, truncation errors will exist in some restricted areas, especially in the regions of high values of the mean climatological ocean current speeds.

The analysis carried out in this paper does not include the part of the advection due to anomalies of wind-driven currents, which depend on the anomalies of surface wind. Therefore, these have been taken as zero, in agreement with the basic assumption in the model considered here, that normal conditions exist in the atmosphere.

Another alternative for determining the permissible time steps is to use the method outline by D. Richtmyer and K.W. Morton (1967).

In conclusion, it can be stated that this study suggests that the time steps have to be shortened when one deals with a more complete forecasting equation. The above results provide a theoretical basis for the choice of the time steps, but due to the simplifying assumptions, they represent only a rough guide that must be confirmed by the numerical experiments.

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