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S–H TYPE WAVES IN A SANDWICHED SPHERICAL MODEL

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RESUMEN

Se estudia la propagación de la onda S-H en una capa esférica heterogénea (la densidad ρ y el módulo de rigidez μ varían inversamente con el cuadrado de la distancia radial (r) desde el centro, de manera tal, que la velocidad de la onda de cizalle es constante) contenida entre una corteza esférica homogénea y un núcleo esférico homogéneo. Se deducen las ecuaciones de periodo y, lo apropiado de su uso se establece mediante resultados numéricos considerando el núcleo central como (1) sólido y (2) fluido.

ABSTRACT

Propagation of S-H type waves in a heterogeneous spherical layer (the density, ρ, and modulus of rigidity μ, vary inversely as the square of the radial distance, r, as measured from the center, such that shear wave velocity is constant) are held between a homogeneous spherical crust and a homogeneous spherical core, the period equations are deduced and their propriety is established by numerical results considering the central core is (1) solid, and (2) fluid.

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INTRODUCTION

Dutta (1973) discussed the propagation of S-H type waves in a heterogeneous spherical earth model compressed between a thin homogeneous crust and a rigid core. The variation of rigidity and density in the model were taken respectively as $\mu_2\ (r/b)^{-m}$ and $\rho_2\ (r/b)^{-m}\ (\vartheta r^{\vartheta-1})^2$, where b is the radius of the rigid core and m and ϑ are arbitrary constants. The period equation was deduced, but the numerical results were obtained as if there were no crust.

In this paper, the model considered is as follows:

(i) the central core is of radius a_1;
(ii) the sandwiched layer lies within $a_1 \leqslant r \leqslant a_2$; and
(iii) the crust lies within the space $a_2 \leqslant r \leqslant a_3$.

The displacements, densities and rigidities are considered respectively as:

(i) ϑ_1, ρ_1, and μ_1, for the homogeneous central core;
(ii) ϑ_2, $\rho_2/\rho_{02} = (a_1/r)^2 = \mu_2/\mu_{02}$ for the heterogeneous sandwiched layer which has the constant shearwave velocity of $\sqrt{\mu_{02}/\rho_{02}}$; and
(iii) ϑ_3, ρ_3, and μ_3 for the homogeneous crustal layer.

The period equations have been deduced by assuming first that the central core is a homogeneous solid sphere, and second that it is a homogeneous fluid sphere. The data are based on the $B_2\ (2)^6$ model (Bullen, 1975, page 254), which divides the interior space of the earth into nine regions. Region A is considered as the crustal layer; the space corresponding to the regions B, C, and D, are defined as the sandwiched model layer; and the central core is defined as the space extending from region E to region G. The same data have been used for both the solid and fluid states of the central core. Numerical calculations have been made with the aid of the *Tables of Functions*. (Jahnkee and Emde, 1945).

FIELD EQUATIONS

Taking the center of the spherical model as the origin, let (r, σ, ϕ) be the spherical polar co-ordinates of any point within the sphere. For S-H type dispersion in a spherical heterogeneous layer model, where both the modulus of rigidity, μ, and density, ρ, are functions of the distance r from the center of the spherical model, the equation of motion is given as:

$$\left(\frac{\partial^2 \vartheta}{\partial r^2} + \frac{2}{r}\frac{\partial \vartheta}{\partial r} - \frac{2}{r^2}\vartheta\right) + \frac{1}{\mu}\frac{d\mu}{dr}\left(\frac{\partial \vartheta}{\partial r} - \frac{\vartheta}{r}\right)$$

$$+ \frac{1}{r^2}\left\{\frac{\partial^2 \vartheta}{\partial \theta^2} + \cot\theta\frac{\partial \vartheta}{\partial \theta} + (1 - \cot^2\theta)\vartheta\right\}$$

$$= \frac{\rho}{\mu}\frac{\partial^2 \vartheta}{\partial t^2} \tag{1}$$

where ϑ is the displacement in the medium under consideration. The solution for equation (1) is sought for each of the three media in the form

$$\vartheta = R(r)\,\Theta(\theta)\,e^{i\omega t} \tag{2}$$

where ω is the angular frequency of the disturbances.

For spherical homogeneous layer model, $\frac{d\mu}{dr} = 0$ in equation (1). Then substituting equation (2) into equation (1), using the method of separation of variables and choosing (n + 2) (n − 1) as the separation constant the following equations are obtained for the crust:

$$\frac{d^2 R}{dr^2} + \frac{2}{r}\frac{dR}{dr} + R\left\{\frac{\omega^2}{\beta_3^2} - \frac{n^2+n}{r^2}\right\} = 0$$

$$\text{and } \frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta} + (1-\cot^2\theta) + (n+2)(n-1)\,\Theta = 0 \qquad (3)$$

$$\text{where: } \beta_3^2 = \frac{\mu_3}{\rho_3}$$

thus the displacement, ϑ_3, within the homogeneous crust may be written:

$$\vartheta_3 = \frac{1}{\sqrt{r}}\left[A J_{n+1/2}(\omega/\beta_3^r) + B N_{n+1/2}(\omega\, r/\beta_3)\right] \times \frac{d}{d\Theta}\left\{P_n(\cos\theta)\right\} e^{i\omega t} \qquad (4)$$

Similarly, the displacement, ϑ_1, within the central homogeneous core can be conveniently taken as:

$$\vartheta_1 = \frac{C}{\sqrt{r}} J_{n+1/2}(\omega r/\beta_1^2)\, \frac{d}{d\theta}\left\{P_n(\cos\Theta)\right\} e^{i\omega t} \qquad (5)$$

$$\text{where } \beta_1^2 = \frac{\mu_1}{\rho_1}$$

For the sandwiched heterogeneous layer, equation (1) and equation (2) lead to:

$$\frac{d^2 R}{dr^2} + R\left\{\frac{\omega^2}{\beta_2^2} - \frac{(n+2)(n-1)}{r^2}\right\} = 0$$

and $$\frac{d^2\Theta}{d\theta^2} + \cot\theta\,\frac{d\Theta}{d\theta} + \left\{(1-\cot\theta) + (n+2)(n-1)\right\}\Theta = 0 \tag{6}$$

where $\beta_2^2 = \frac{\mu_{02}}{\rho_{02}}$

Therefore, the displacement ϑ_2 within this layer is given as:

$$\vartheta_2 = \sqrt{r}\,[DJ_p(\frac{\omega r}{\beta_2}) + EN_p(\frac{\omega r}{\beta_2})]\,\frac{d}{d\theta}\left\{P_n\cos\theta)\right\}\left(e^{i\omega t}\right) \tag{7}$$

where $p = \sqrt{n^2 + n - 7/4}$

FORMULATION OF THE PROBLEM

When a heterogeneous spherical layer is located between a homogeneous spherical crust and a homogeneous solid central core, the boundary conditions for S-H wave propagation are:

$$\text{(i)}\quad \vartheta_2 = \vartheta_1;\ \mu_2\left(\frac{\partial\vartheta_2}{\partial r} - \frac{\vartheta_2}{r}\right) = \mu_1\left(\frac{\partial\vartheta_1}{\partial r} - \frac{\vartheta_1}{r}\right), \text{ for } r = a_1$$

$$\text{(ii)}\quad \vartheta_2 = \vartheta_3;\ \mu_2\left(\frac{\partial\vartheta_2}{\partial r} - \frac{\vartheta_2}{r}\right) = \mu_3\left(\frac{\partial\vartheta_3}{\partial r} - \frac{\vartheta_3}{r}\right), \text{ for } r = a_2 \tag{8}$$

$$\text{(iii)}\quad \frac{\partial\vartheta_3}{\partial r} - \frac{\vartheta_3}{r} = 0, \text{ for } r = a_3.$$

Employing equations (4), (5) and (7) together with the above conditions lead to the period equation (9).

If the central core is assumed to be in fluid state the shearing stress, $\frac{\partial\vartheta_2}{\partial r} - \frac{\partial\vartheta_2}{\partial r} = 0$ for $r = a_1$. Substitution of this shearing stress for the second condition in (8) and retaining the other conditions leads to another period equation. This new period equation is similar to equation. (9), differing only on the third element of the second row which is zero.

Period equation (9)

$$\left[\text{taking } \frac{\omega}{\beta_2} a_3 = X\right]$$

$$\begin{vmatrix}
J_p\left(\frac{a_1}{a_3}x\right) & N_p\left(\frac{a_1}{a_3}x\right) & J_{n+\frac{1}{2}}\left(\frac{a_1}{a_3}\cdot\frac{\beta_2}{\beta_1}x\right) & 0 & 0 \\
\frac{\mu_{02}}{\mu_1}\left\{J'_p\left(\frac{a_1}{a_3}x\right)\left(\frac{a_1}{a_3}x\right) - \frac{1}{2}J_p\left(\frac{a_1}{a_3}x\right)\right\} & \frac{U_{02}}{U_1}\left\{\frac{a_1}{a_3}x N'_p\left(\frac{a_1}{a_3}x\right) - \frac{1}{2}N_p\left(\frac{a_1}{a_3}x\right)\right\} & \left\{\frac{a_1}{a_3}\frac{\beta_2}{\beta_1}x J_{n+\frac{1}{2}}\left(\frac{a_1}{a_3}\frac{\beta_2}{\beta_1}x\right) - \frac{3}{2}J_{n+\frac{1}{2}}\left(\frac{a_1}{a_3}\cdot\frac{\beta_2}{\beta_1}x\right)\right\} & 0 & 0 \\
\frac{a_2}{a_1}J_p\left(\frac{a_2}{a_3}x\right) & \frac{a_2}{a_1}N_p\left(\frac{a_2}{a_3}x\right) & 0 & J_{n+\frac{1}{2}}\left(\frac{\beta_2}{\beta_3}\cdot\frac{a_2}{a_3}x\right) & N_{n+\frac{1}{2}}\left(\frac{\beta_2}{\beta_3}\frac{a_2}{a_3}x\right) \\
\frac{a_1}{a_2}\frac{\mu_{02}}{\mu_3}\left\{\frac{a_2}{a_3}x J'_p\left(\frac{a_2}{a_3}x\right) - \frac{1}{2}J_p\left(\frac{a_2}{a_3}x\right)\right\} & \frac{a_1}{a_2}\frac{U_{02}}{U_3}\left\{\frac{a_2}{a_3}x N'_p\left(\frac{a_2}{a_3}x\right) - \frac{1}{2}N_p\left(\frac{a_2}{a_3}x\right)\right\} & 0 & \left\{\frac{\beta_2}{\beta_3}\frac{a_2}{a_3}x J'_{n+\frac{1}{2}}\left(\frac{\beta_2}{\beta_3}\frac{a_2}{a_3}x\right) - \frac{3}{2}J_{n+\frac{1}{2}}\left(\frac{\beta_2}{\beta_3}\frac{a_2}{a_3}x\right)\right\} & \left\{\frac{\beta_2}{\beta_3}\frac{a_2}{a_3}x N'_{n+\frac{1}{2}}\left(\frac{\beta_2}{\beta_3}\frac{a_2}{a_3}x\right) - \frac{3}{2}N_{n+\frac{1}{2}}\left(\frac{\beta_2}{\beta_3}\cdot\frac{a_2}{a_3}x\right)\right\} \\
0 & 0 & 0 & \left\{\frac{\beta_2}{\beta_3}x J'_{n+\frac{1}{2}}\left(\frac{\beta_2}{\beta_3}x\right) - \frac{3}{2}J_{n+\frac{1}{2}}\left(\frac{\beta_2}{\beta_3}x\right)\right\} & \left\{\frac{\beta_2}{\beta_3}x N'_{n+\frac{1}{2}}\left(\frac{\beta_2}{\beta_3}x\right) - \frac{3}{2}N_{n+\frac{1}{2}}\left(\frac{\beta_2}{\beta_3}x\right)\right\}
\end{vmatrix} = 0$$

NUMERICAL CALCULATIONS

The earth is defined as a sphere of radius 6,371 km, divided into three layers:

(i) Crustal layer: 6,371 km $\geqslant r \geqslant$ 6.338 km;
(ii) Sandwiched layer: 6,338 km $\geqslant r \geqslant$ 3,485 km; and
(iii) Central core: 3,485 km $\geqslant r \geqslant 0$.

Thus, a_1 = 3,485 km; a_2 = 6,338 km; and a_3 = 6,371 km.

A relevant portion of the data presented by Bullen[2] in the B_2 model are quoted below:

Table 1

Region	Depth (km)	ρ (gm/cm^3)	μ (10^{12} dynes/cm^2)
A	33	3.32	0.63
D"	2886	5.69	3.04
E'	2886	9.95	0

E"	4710	12.3	0
F	4710	12.3	0.53
G	6371	13.03	1.11

Based on the context of the data, the following values of the elastic constants have been assumed for numerical calculations:

(i) ρ_1 = 12.3 gm/cm^3, corresponding to a depth of 4,710 km for region F.

(ii) ρ_{02} = 5.69 gm/cm^3 (according to the assumed law of density variation this value may be approximately 10.98 gm/cm^3 if the density at the interface with the crust is assumed to be 3.32 gm/cm^3. It should be noted that the density has a jump-discontinuity at a depth of 2,886 km. At this same depth, on the surface of the core, the density is 9.95 gm/cm^3.);

(iii) ρ_3 = 3.32 gm/cm^3;

(iv) $\mu_1 = 0.53 \times 10^{12}$ dynes/cm^2; and

(v) $\mu_{02} = 3.04 \times 10^{12}$ dynes/cm^2 (according to the assumed law of rigidity variation, this value is approximately equal to 2.0835×10^{12} dynes/cm^2 considering that $\mu_1 = 0.53 \times 10^{12}$ dynes/cm^2 on the surface of the sandwiched layer. It is of interest to note that on the surface of the core the rigidity vanishes).

It is noteworthy that the discrepancies in the data regarding the assumed values of ρ_{02} and μ_{02} honor vary little the laws of variation of the density and the rigidity modulus; simultaneously, the assumption that the shear wave velocity is constant withing the intermediate layer is loosely adhered too.

Thus, given n = 1 whence p = $^1\!/_2$, the period equation (9) which assumes the core is in a solid state) has one of its roots at approximately $\chi = \omega a_3/\beta_2 \doteq 3.87547$. The revised period equation for the fluid core possesses roots at approximately $\chi \doteq 2.3439757$, $x \doteq 4.00387$, and $x = 5.648565$, etc.

In the two layared problem of the propagation of S-H waves in a spherical model, Dutta (1973) assumed n = 1, p = 1, and $\rho/\rho_2 = (b/r)^5 = \mu/\mu_2$ for the heterogeneous upper stratum such that the shear wave velocity within the mantle is constant. Moreover he assumed:

(i) the radius of the earth = a = 6,371 km, and

(ii) the radius of the rigid core = b = .548 a.

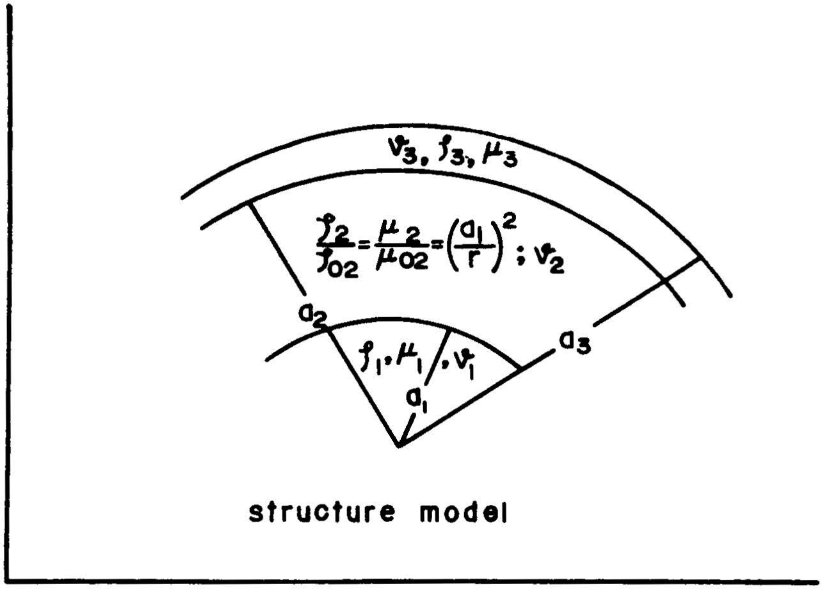
Using these values he found x = 3.899.

Corresponding to a given value of n, there are generally two possible values for p, equal in magnitude, but opposite in sign. For a real value for p and x, the phase velocity, as a function of the wave number is represented by a rectangular hyperbola having the axes (wave number and phase velocity) as the asymptotes, x = 0 is also a root of equation (9). This implies that the family of phase velocity curves represented by a family of rectangular hyperbolas is limited by the axes proposed. There is no group velocity withing the mantle.

DISCUSSION

The assumption that the crust is homogeneous is generally speaking a valid one provided that it is of small depth in comparison with the radius of the earth. A close examination of the data reveals that the rates of variation of the density and rigidity modulus at depths below the earth's surface are far from similar. The rate of variation is higher for the rigidity modulus than that of the density, and the shear wave velocity (μ_2/ρ_2) gradually increases until a depth of 2,886 km is reached. Beyond this depth, the density jumps abruptly, increasing by more than 74.868 per cent, and the rigidity modulus becomes zero at the same depth on the surface of the assumed core. Thereafter, the density increases with depth, but the earth's materials between the dephts of 2,886 km and 4,710 km have a negligible modulus of rigidity. Beyond the depth of 4,710 km, again increases with depth. Thus, the central core consists of two parts possessing entirelly different characteristics. For the sake of simplicity the core is assumed to be homogeneous, and its behavior is similar to the core at a depth of 4,710 km. It is almost certain that the core is not rigid, but one of high density with low regidity, and that the shear wave velocity is very slight. Based on these facts, one may conclude that the core is either semi-solid or fluid. The density and the rigidity modulus of the sandwiched layer, between the depths of 33 km and 2,886 km increase with depth, but at different rates. It would be difficult to encounter a definite law for either the density of the rigidity modulus variations which would be compatible with the results based on the data presented here. it is appropriate to mention that the data can not assert a fit along all of the radial directions within the earth-model. The earth many be regarded as a mystery in that it possesses different material properties in different regions of small or large stretches. Thus, it would not be possible to locate a region where the assumptions regarding the density and rigidity modulus would approximate reality.

Lastly, I express my thanks to Dr. S. Dutta for his valuable suggestions in preparing this paper.


v_3, ρ_3, μ_3
$\frac{\rho_2}{\rho_{02}} = \frac{\mu_2}{\mu_{02}} = \left(\frac{a_1}{r}\right)^2 ; v_2$
a_2
ρ_1, μ_1, v_1
a_1
a_3
structure model

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