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An incomplete factorization technique for fast numerical solution of steady-state groundwater flow problems

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RESUMEN

Se presenta una técnica nueva iterativa para resolver ecuaciones de diferencias finitas en problemas estables de hidrodinámica subterránea.

PALABRAS CLAVE: Factorización incompleta, problema estable, hidrodinámica subterránea.

ABSTRACT

The effectiveness of a new iterative technique for solving finite-difference equations is demonstrated for steady-state problems of hydrodynamics of groundwater flow.

KEY WORDS: Incomplete factorization, steady-state problem, underground hydrodynamics.

INTRODUCTION

Problems of a moving underground liquid were subjects of many articles in the last thirty years. Computer progress directs attention to numerical solutions of such problems, and numerical simulations of transient processes have been highly developed. But for steady state problems the numerical solutions were uneconomical and not very attractive. The more the boundary conditions differed from Dirichlet condition, the less the numerical solution was economical. For some of steady-state problems analytical solutions were developed, but most of them are not simple for numerical realization.

Ginkin (1977) and Sabinin (1980, 1985) have developed a variant of the incomplete factorization method (Buleev, 1970) which permits the effective numerical solution of steady state problems even for near Neumann boundary conditions and quasi-linear equations. This fast iterative technique is here improved with a new choice of iteration parameters, and is applied to some steady-state problems of groundwater flow.

1. THE 2-D PROBLEM OF FLUID FLOW TOWARDS A WELL

Consider a layer of soil with impermeable upper and lower boundaries, with anisotropic conductivity, and with a well perforated in this layer. An axis-symmetric fluid movement can be described by the Laplace equation in cylindrical co-ordinates:

$$\frac{K_r}{r}\frac{\partial}{\partial r}\left(r\frac{\partial H}{\partial r}\right) + K_z\frac{\partial^2 H}{\partial z^2} = 0, \tag{1}$$

where H is the hydraulic head, and K_r and K_z are the hydraulic conductivities in the horizontal and vertical directions.

The boundary-value conditions are as follows:

$H(r,z) = H_0$ at $r = r_0$ and $0 \le z_0 \le z \le z_1 \le Z$; $\frac{\partial H}{\partial r} = 0$ at $r=r_0$ and $z < z_0$, $z > z_1$; $H(r,z) = H_1$ at $r = R$ and $0 \le z \le Z$; and $\frac{\partial H}{\partial z} = 0$ at $z = 0$, and at $z = Z$, for all r; - (2)

where r_0 is the radius of the well, R is the prescribed radial distant boundary, Z is the thickness of the layer, and z_0 and z_1 are the vertical coordinates of edges of the perforated filter of the well.

The solution of the steady-state problem (1)-(2) is found by the finite-difference method. The finite-difference solution will be obtained by an iterative technique of incomplete factorization of implicit type. For the best performance of the iterative method the equation (1) is to be first transformed by the substitution $y = r^2/4$ into:

$$K_r\frac{\partial}{\partial y}\left(y\frac{\partial H}{\partial y}\right) + K_z\frac{\partial^2 H}{\partial z^2} = 0. \tag{3}$$

For the solution of equation (3), we use a uniform grid in the (y, z) plane and a 5-point template. Thus, the finite-difference approximation of (3) is

$$\frac{K_r}{2\bar{h}_i h_i}[(H_{i+1,j}-H_{ij})(y_i+y_{i+1})-(H_{i,j}-H_{i-1,j})(y_i+y_{i-1})]+$$
$$\frac{K_z}{\bar{h}_j h_j}(H_{i,j+1}-2H_{ij}+H_{i,j-1})=0\,,\quad i=0,\ldots,I\,,\ j=0,\ldots,J. \tag{4}$$

Here h_i and h_j are the constant steps in the y and z directions; $\bar{h}_i = h_i/2$ for i=0 and i=I, and $\bar{h}_i = h_i$ for other values of i; $\bar{h}_j = h_j/2$ for j=0 and j=J, and $\bar{h}_j = h_j$ for other values of j.

The system (4) together with boundary conditions (2) has the following operator form:

$$AH_{ij} \equiv a_{ij}H_{i-1,j}+b_{ij}H_{i,j-1}+c_{ij}H_{i+1,j}+d_{ij}H_{i,j+1}-e_{ij}H_{ij}=-f_{ij}. \tag{5}$$

The iterative process for the solution of (5) is as follows:

$$LU(H_{ij}^{s+1}-H_{ij}^{s})=AH_{ij}^{s}+f_{ij}^{s}\,,\quad s=0,1,\ldots \tag{6}$$

The operators L and U have the form

$$Lv_{ij} \equiv v_{ij}-\alpha_{ij}v_{i-1,j}\,, \tag{7}$$
$$Uu_{ij} \equiv \gamma_{ij}u_{ij}-\beta_{ij}u_{i,j-1}-\delta_{ij}u_{i,j+1}-\xi_{ij}u_{i+1,j}\,. \tag{8}$$

The process (6) can be written as a sequence of three equations

$$Lv_{ij}=AH_{ij}^{s}+f_{ij}^{s}\,,\ Uu_{ij}=v_{ij}\,,\ H_{ij}^{s+1}=H_{ij}^{s}+u_{ij}\,. \tag{9}$$

It is necessary to add to (9) the equations for definition of the coefficients of the operators L and U. They are found from a solution of operator equation

$$LU=-A-F+B\,, \tag{10}$$

where operator $F=(\partial f/\partial H)_{ij}^{s}$ if f depends on H, following Newton iteration method applied to f. For the problem (1)-(2), $F \equiv 0$.

As a template of a product LU has two additional points (i-1, j-1) and (i-1, j+1), in comparing with a template of A, as a template of operator B should compensate it. Over that, the operator B should have a template which provides best convergence properties for the iteration process. Best choice is a use of a following interpolation equation (Sabinin, 1980) in building operator B:

$$H_{i-1,j\pm1}+\omega H_{ij}=H_{i,j\pm1}+\omega H_{i-1,j}\,,$$

where ω is an iteration parameter.

It yields a definition of B, as follows:

$$\begin{aligned}BH_{ij} \equiv\ & \alpha_{ij}\beta_{i-1,j}[H_{i-1,j-1}-H_{i,j-1}+\omega(H_{ij}-H_{i-1,j})]+\\ & \alpha_{ij}\delta_{i-1,j}[H_{i-1,j+1}-H_{i,j+1}+\omega(H_{ij}-H_{i-1,j})]\,.\end{aligned} \tag{11}$$

After solving (10) for each point of the template we may derive:

$$\begin{aligned}&\alpha_{ij}=a_{ij}/[\gamma-\omega_s(\beta'+\delta')]_{i-1,j}\,,\ \xi_{ij}=c_{ij}\,,\\ &\beta_{ij}=b_{ij}+\alpha_{ij}\beta'_{i-1,j}\,,\ \delta_{ij}=d_{ij}+\alpha_{ij}\,\delta'_{i-1,j}\,,\\ &\gamma_{ij}=e_{ij}-(\partial f/\partial H)_{ij}^{s}-a_{ij}+\alpha_{ij}(\gamma-c)_{i-1,j}\,,\end{aligned} \tag{12}$$

where $\beta'_{ij}=\beta_{ij}$ for all nodes except node (0, j_1+1), in which $\beta'=0$; and $\delta'_{ij}=\delta_{ij}$ for all nodes except node (0, j_0-1), in which $\delta'=0$; and j_0 and j_1 are the grid nodes corresponding to z_0 and z_1.

In general case a, b, c, d, e, and f may depend on the solution H. In this case, it is implied that their values are taken from the iteration s in (12).

Variant (7), (8), (12) of definition of operators L and U is suitable for problems with a Dirichlet boundary condition presented at the boundary $x = X$ (that is $i = I$). For the general form of the technique, see the Appendix.

For iteration parameter ω_s in (12), we may use the following cyclic set of values (Sabinin, 1985):

$$\begin{aligned}&\omega_0=1\,;\ \omega_s=1-2\sqrt{\eta}\,q^{(1-2\sigma)/4}\,\frac{1+q^{\sigma}+q^{2-\sigma}}{1+q^{1+\sigma}+q^{1-\sigma}}\,,\ s=1,\ldots,l\,;\\ &\omega_s=\omega_{s-l}\,,\ s>l\,,\end{aligned} \tag{13}$$

where $\eta=\sin^2(\frac{\pi}{2J})$, $q=\eta^2\,(1+\eta^2/2)/16$, $\sigma=(2s\text{-}1)/(2l)$. It is derived by analogy with a Jordan set of iteration parameters for Alternating Direction Implicit method (see Samarsky and Nikolaev, 1978, and Wachspress, 1962).

Here a more effective set of values is suggested, which differs from (13) by its definition of η, and ω_s, and by dependence on i. If we express (13) in the form ω_s = 1 -2R_s, the new equation for ω_s will be

$$\omega_s=1-1.5\ [1+\frac{c}{4(i+1)}]R_s \text{ for } i<I-1\text{, and}$$

$$\omega_s = 1 - 1.5(2 + \frac{1}{2I})R_s \text{ for } i = I - 1. \qquad (13a)$$

In the definition of R_s, $\eta = \sin^2(\frac{\pi}{2J_i})$ for $i = I$-1, and

$\eta = \sin^2(\frac{\pi}{2J_i}) + \frac{c}{2(i+1)}\sin(\frac{\pi}{2J_i})$ for $i < I$ - 1. Here J_i is the number of steps h_j covering the area along the line i, and $c = 0.1\sqrt{J_i}$.

This set of parameters is not optimal, but it yields fast convergence for different types of problems, and therefore it is closer to the universal one than (13).

The iterative process (7)-(9), (12), (13a) is terminated, and the solution is achieved when condition

$$\max_{ij}\left|AH_{ij}^{s}\right| / \max_{ij}\left|AH_{ij}^{0}\right| \leq \varepsilon \qquad (14)$$

is fulfilled, where ε is a some prescribed small value.

This technique is fast for different boundary conditions and for non-rectangular areas, as will be shown below.

Formulas (7)-(8), (12) can be called as "i-directed". We may write the "j-directed" formulas similarly, and apply them to problems with a Dirichlet boundary condition at the boundary $y = Y$. If Dirichlet boundary conditions were presented at both x=X and y=Y, then the "i-directed" formulas would be better in a case $a_{ij} >> b_{ij}$, and the "j-directed" - when $b_{ij} >> a_{ij}$ (Ginkin, 1977).

In Figure 1, lines of equal heads are shown for the following set of parameters. Initially, K_r=K_z=1, because the ratio K_z/K_r can be inserted into the definition of the vertical size of the area Z. In this variant Z=20, R=100, and r_0=1. The finite-difference grid sizes were I=J=100. The lower edge of the filter was at j=30, and the upper edge was at j=70.

The prescribed heads at the filter and at the right boundary were: H_0 = -10, and H_1 = 0. The initial head for the start of the iterative process was a constant and equal to 0.5(H_0+H_1). In Figure 1, the 20 lines of equally spaced values of the head between H_0 and H_1 are shown.

An accuracy ε=0.00001 is achieved at iteration 31.

The variant in Figure 2 differs from Figure 1 by the position and size of the filter and by the size of the area Z=80. The lower edge of the filter was at j = 0, and the upper edge was at j = 20. An accuracy ε=0.00001 is achieved here at iteration 12.

As seen from these examples, the proposed numerical solution is economical and can be applied for describing a fluid flow near wells.

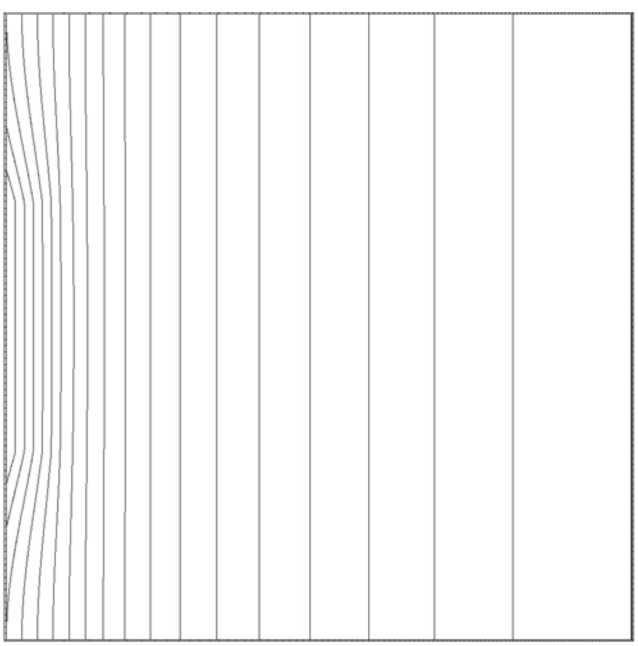

Fig. 1. Equidistant lines of equal head for the centered position of the filter and a stretched area.

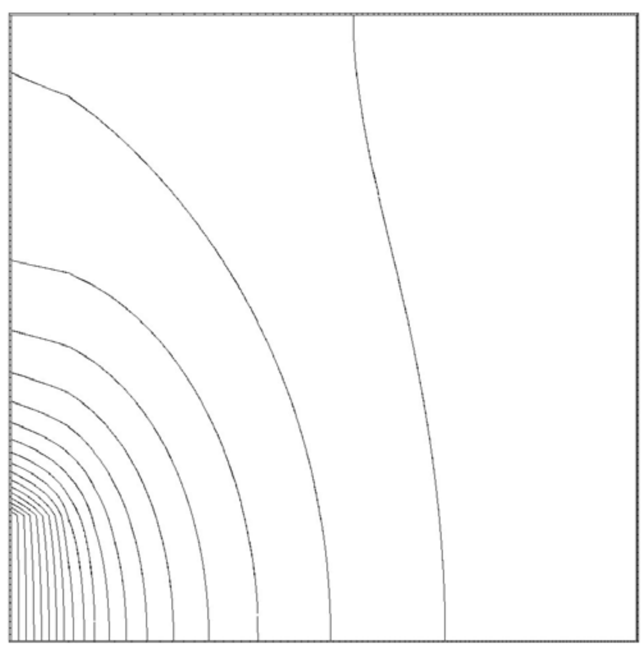

Fig. 2. Equidistant lines of equal head for the corner position of the filter and a near-square area.

2. GROUNDWATER FLOW BETWEEN ADJACENT VALLEY FLOOR AND WATERSHED

Following Toth (1962), the problem can be described by a 2D boundary-value problem in the rectangle area (*0,0*)-(*X,Z*) for the Laplace equation

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial z^2} = 0 \tag{15}$$

with the boundary conditions:

$$H = Z + cx \text{ at } z = Z; \frac{\partial H}{\partial z} = 0 \text{ at } z = 0; \frac{\partial H}{\partial x} = 0 \text{ at } x = 0, \text{and at } x = X. \tag{16}$$

The problem (15)-(16) is solved by the general method of contrary directions (Sabinin, 1985), described in the Appendix.

An example of calculations is presented in Figure 3. In this variant the area has dimensions $X = 1$ and $Z = 1$, inclination $c = 0.01$ in (16), and a uniform finite-difference grid has dimensions $I = J = 500$. The initial head is equal to Z. A numerical solution with accuracy ε=0.00001 is obtained after 6 iterations.

The analytical solution by Toth (1962) agrees with the numerical solution within a maximum relative error of 0.000004.

A variant which differs from Figure 3 by a dimension $X = 10$ converges to the same accuracy after two iterations, and the solution agrees with the analytical solution of Toth (1962) within a maximum relative error of 0.00065.

3. GROUNDWATER FLOW FROM A SURFACE TO A DRAIN

This is the same problem of equation (15) in the rectangle, with the following boundary value conditions:

$$H = H_1 \text{ at } z = Z; \ \frac{\partial H}{\partial z} = 0 \text{ at } z = 0; \ \frac{\partial H}{\partial x} = 0 \text{ at } x = 0; \ \frac{\partial H}{\partial x} = 0$$
$$\text{at } x = X \text{ and } z > z_0 > 0; \tag{17}$$

and $H = H_0$ at $x = X$ and $z \le z_0$.

The problem (15), (17) is very similar to (1), (2). This problem may be solved by the technique found in the Appendix.

An example of application is found below. The variant of Figure 4 has the following parameters: X=Z=1, H_0=0, H_1=5, I=J=500, and z_0=Z/1000; that is, only one boundary node of the grid belongs to the drain section. The initial distribution of the head is H_1.

A numerical solution with iteration accuracy ε=0.00001 is obtained after 22 iterations.

A variant which differs from Figure 4 by the dimension X=5 converges to the same accuracy after 19 iterations.

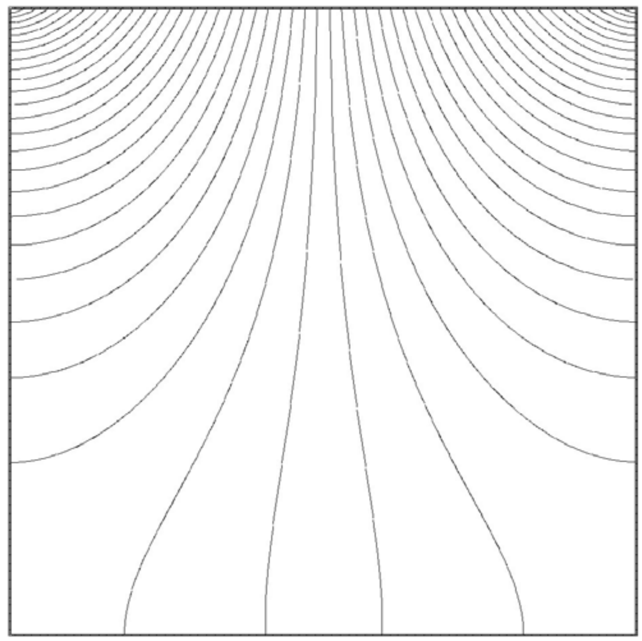

Fig. 3. Equidistant lines of equal head for a square area.

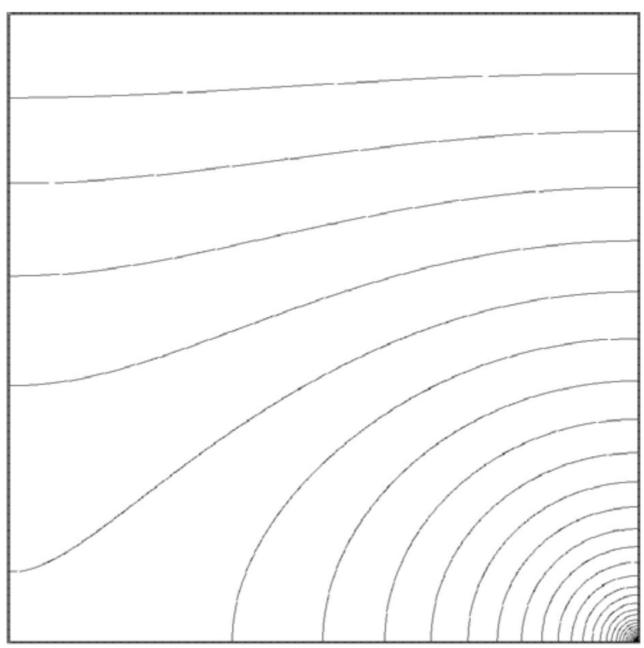

Fig. 4. Equidistant lines of equal head for a square area.

4. THE POISSON EQUATION IN A CIRCULAR AREA

As an example of application of the iterative technique to a non-rectangular area, the Neumann-type boundary value problem for the Poisson equation is considered for an area which is a step-wise approximation to a circle:

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial z^2} = 2(x^2 + z^2), \qquad (18)$$

$\frac{\partial H}{\partial x} = 2xz^2$ at every vertical segment of the boundary, and $\frac{\partial H}{\partial z} = 2x^2 z$ at every horizontal segment of the boundary. Thus, the solution of this boundary-value problem is $H=x^2z^2$, and at one boundary point shared with the circumscribed square, the boundary value H is set to this solution.

A five-point finite-difference approximation of type (5) to problem (18) is a so-called precise scheme, and therefore the finite-difference solution is equal to the solution of (18). For the condition of termination of the iterative process, we may use $\max_{ij}\left|H_{ij}^s - H\right| / \max_{ij}\left|H_{ij}^0 - H\right| \le \varepsilon$ instead of condition (14).

For the solution, the technique in the Appendix was applied. For testing, the mostly inconvenient initial distribution was used: H_{ij}^0=H+1 for 2(i+j)<I+J, and H_{ij}^0=H-1 for 2(i+j)≥I+J. For a maximum grid size $I = J = 500$ the solution with iteration accuracy $\varepsilon = 0.00001$ was obtained after 39 iterations. However, condition (14) was achieved after 25 iterations.

In Figure 5, forty equidistant lines of equal H from solution of this problem are shown.

5. OIL-OVER-WATER FLOWING TOWARD A HORIZONTAL DRAIN

Guo *et al.* (1992) provided the analytical solution of this linear problem. For numerical modeling, a possible approach is a similar non-linear 2D boundary-value problem in the rectangular area (0,0)-(X,Z) for a quasi-linear Laplace equation of the head H:

$$\frac{\partial}{\partial x}\left(K\frac{\partial H}{\partial x}\right) + \frac{\partial}{\partial z}\left(K\frac{\partial H}{\partial z}\right) = 0, \qquad (19)$$

where $H = \rho gz+P$, P is the capillary pressure of oil, and

$$\begin{aligned} K &= 1/[1 + (P_w - P)^m / P_0^m] \quad for \quad P \le P_w, \\ K &= 1 \quad for \quad P \ge P_w, \end{aligned} \qquad (20)$$

with the restriction $K \ge 0.00001$; and the capillary pressure of water is $P_w = -\rho_w gz$; with boundary conditions:

$P = P_E$ at x=X, at the point drain; $\frac{\partial H}{\partial z} = 0$ at z=0, and z=Z;

$\frac{\partial H}{\partial x}$=$q/(2Z)$ at x=0; (21)

and $\frac{\partial H}{\partial x} = 0$ at x=X except at the drain.

The problem (19)-(21) is a near Neumann boundary-value problem for a non-linear equation. Such problems are most uneconomical for a numerical solution. A choice of an initial iteration and parameters of non-linearity (for our problem, P_0) influence on convergence of iterative methods significantly. Usual practice is that for some sets of parameters of the problem an iterative solution may not exist.

The five-point finite-difference scheme for (19) is written as follows:

$$AH_{ij}^s \equiv \frac{(H_{i+1,j}^s - H_{ij}^s)(K_{ij}^s + K_{i+1,j}^s) - (H_{i,j}^s - H_{i-1,j}^s)(K_{ij}^s + K_{i-1,j}^s)}{2\bar{h}_i h_i}$$
$$+ \frac{(H_{i,j+1}^s - H_{ij}^s)(K_{ij}^s + K_{i,j+1}^s) - (H_{i,j}^s - H_{i,j-1}^s)(K_{ij}^s + K_{i,j-1}^s)}{2\bar{h}_j h_j} = 0. \qquad (22)$$

The iterative method (7)-(9), (12), (13a) is applied to equation (22) with one modification. For non-linear problems, it is better to use only positive values of the set (13a) and to include the value ω=1 in all cycles, and not only in the first.

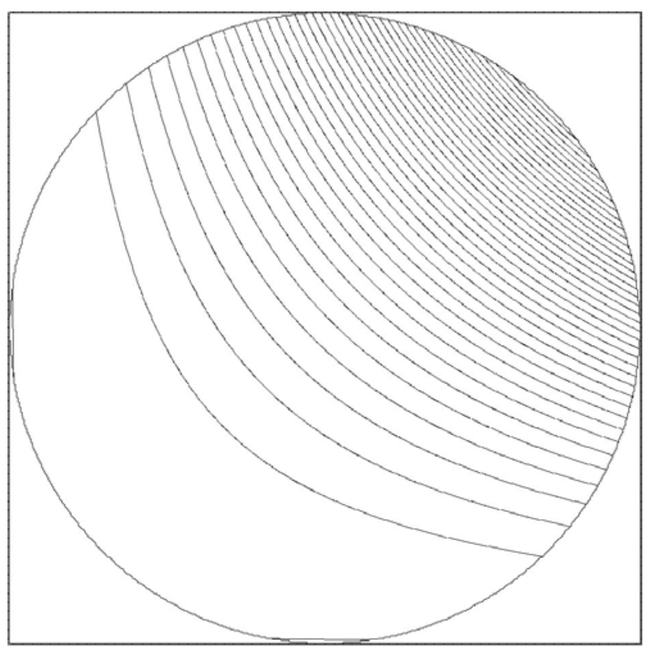

Fig. 5. Equidistant lines of equal head for a circular area.

Figure 6 shows the variant corresponding to the example of Guo *et al*. (1992).

The dimension of the area is 320x37 m, the grid size is 200x200, the influx is q=35598m^2/s, for oil ρg=7693, for water $\rho_w g$=9225, P_0=5000, and P_E=-240000, in the international system C. The drain is located at the grid node j=140. Initial distribution of the head corresponds to P=P_E.

The dotted curve at the lower right is the water/oil interface. The resulting position of the water/oil interface is near one of Guo *et al*. (1992), but differs because of differences in the hydrodynamic models. As it turns out, the water-oil interface P=P_w is very sensitive to the parameters of the model.

The number of iterations for this variant is 100 for ε=0.00001, that is somewhat large. But with increasing P_0 the number of iterations decreases, provided the water/oil interface changes insignificantly. For P_0 = 10 000 the number of iterations is 69, and for P_0=40 000 it is 42. The number of iterations greatly depends on a value P_E. Attempts to use a Newton iteration method for accelerating convergence did not yield better results. But good initial distributions of head improved the convergence significantly. This demonstrates the difficulty of solving the Neumann boundary value problem for non-linear equations.

DISCUSSION AND CONCLUSIONS

The hydrodynamics problems of liquid flow towards wells or underground drains are uneconomical for numerical solutions, especially for steady-state flows, because of the predominant presence of Neumann boundary conditions. Our technique is powerful for reducing computational expense. The main features of the technique and its extension to three dimensions were presented by Sabinin (1985). Some non-linear 2D and 3D groundwater problems were solved with this method (Sabinin, 1980, 1981, 1999).

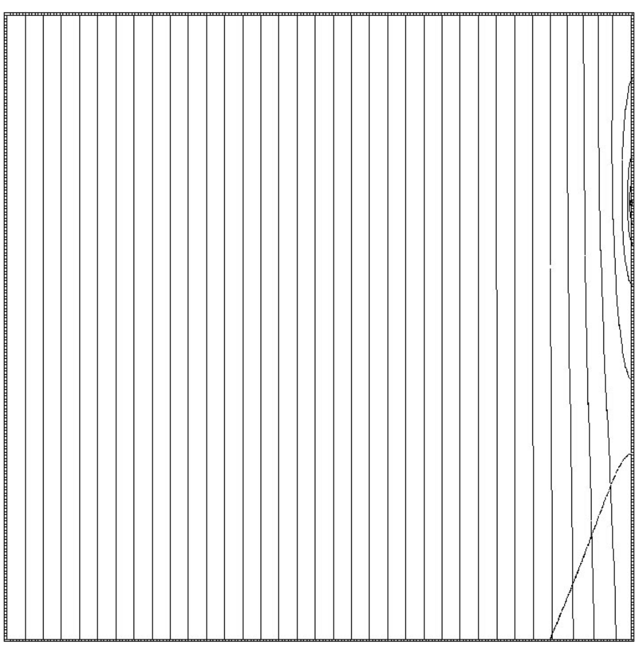

Fig. 6. The 40 equidistant lines of equal head and a position of the water/oil interface.

The technique is at least as fast as the best-known iterative methods (see, for comparison, Samarsky and Nikolaev, 1978, and Sabinin, 1985). It is more flexible in applications to different boundary-value problems.

We have applied the technique to some 2D steady-state problems. The results demonstrate the high effectiveness of this technique for elliptic equations with constant and variable coefficients, and for non-rectangular areas. For quasi-linear elliptic equations, the effectiveness of the technique depends on an initial iteration and a steepness of non-linear coefficients. As shown, the technique may yield sufficiently fast convergence even for near Neumann boundary-value problems.

ACKNOWLEDGEMENT

I consider as a pleasant debt to express gratitude to Mario García who has drawn my attention to the problems 1, 2, 3, and 5.

APPENDIX. GENERAL FORM OF THE TECHNIQUE

In the case of the Dirichlet boundary condition located at a grid node with i=i_0, and 0<i_0<I, the following general form of operators L and U should (or may) be used (Sabinin, 1985).

$$Lv_{ij} \equiv v_{ij} - \alpha_{i-1,j} v_{i-1,j} \quad for \quad i < i_0,$$
$$Lv_{ij} \equiv v_{ij} - \alpha_{i+1,j} v_{i+1,j} \quad for \quad i > i_0,$$
$$Lv_{ij} \equiv v_{ij} - \alpha_{i-1,j} v_{i-1,j} - \alpha_{i+1,j} v_{i+1,j} \quad for \quad i = i_0;$$
$$Uu_{ij} \equiv \gamma_{ij} u_{ij} - \beta_{ij} u_{i,j-1} - \delta_{ij} u_{i,j+1} - c_{ij} u_{i+1,j} \quad for \quad i < i_0,$$
$$Uu_{ij} \equiv \gamma_{ij} u_{ij} - \beta_{ij} u_{i,j-1} - \delta_{ij} u_{i,j+1} - a_{ij} u_{i-1,j} \quad for \quad i > i_0,$$
$$Uu_{ij} \equiv \gamma_{ij} u_{ij} - \beta_{ij} u_{i,j-1} - \delta_{ij} u_{i,j+1} \quad for \quad i = i_0.$$

The equations for definition of the coefficients of the operators L and U are as follows:

$$\alpha_{i-1,j} = a_{ij} / [\gamma - \omega_s (\beta' + \delta')]_{i-1,j},$$
$$\alpha_{i+1,j} = c_{ij} / [\gamma - \omega_s (\beta' + \delta')]_{i+1,j},$$
$$\beta_{ij} = b_{ij} + \alpha_{i-1,j} \beta'_{i-1,j}, \ \delta_{ij} = d_{ij} + \alpha_{i-1,j} \delta'_{i-1,j} \quad for \quad i < i_0;$$
$$\beta_{ij} = b_{ij} + \alpha_{i+1,j} \beta'_{i+1,j}, \ \delta_{ij} = d_{ij} + \alpha_{i+1,j} \delta'_{i+1,j} \quad for \quad i > i_0;$$

$$\beta_{ij} = b_{ij} + \alpha_{i-1,j}\beta'_{i-1,j} + \alpha_{i+1,j}\beta'_{i+1,j}, \text{ and},$$
$$\delta_{ij} = d_{ij} + \alpha_{i-1,j}\delta'_{i-1,j} + \alpha_{i+1,j}\delta'_{i+1,j} \quad for \ i = i_0;$$
$$\gamma_{ij} = e_{ij} - (\partial f/\partial H)^s_{ij} - a_{ij} + \alpha_{i-1,j}(\gamma - c)_{i-1,j} \quad for \ i < i_0,$$
$$\gamma_{ij} = e_{ij} - (\partial f/\partial H)^s_{ij} - c_{ij} + \alpha_{i+1,j}(\gamma - a)_{i+1,j} \quad for \ i > i_0,$$
$$\gamma_{ij} = e_{ij} - (\partial f/\partial H)^s_{ij} - a_{ij} - c_{ij} + \alpha_{i-1,j}(\gamma - c)_{i-1,j} + \alpha_{i+1,j}(\gamma - a)_{i+1,j} \quad for \ i = i_0,$$

where $\beta'_{ij} = \beta_{ij}$ for all nodes except those nodes for which the adjacent (i, j+1) node fulfils the Dirichlet boundary condition, for such nodes $\beta'_{ij} = 0$; and $\delta'_{ij} = \delta_{ij}$ except for the nodes where the adjacent (i, j-1) node fulfils the Dirichlet boundary condition, at such nodes $\delta'_{ij} = 0$.

These equations for coefficients are derived from solution of the operator equation

$$LU = -A - F + B,$$

where operator $F = (\partial f/\partial H)^s_{ij}$.

A template of a product LU has two or four additional points, in comparing with a template of A: (i-1, j-1) and (i-1, j+1) for $i<i_0$, (i+1, j-1) and (i+1, j+1) for $i>i_0$, and both these pairs for $i=i_0$. For this case, operator B is built on a base of an interpolation equation

$$H_{i\pm1,j\pm1} + \omega H_{ij} = H_{i,j\pm1} + \omega H_{i\pm1,j},$$

and has a template shown in Figure 7. It is defined by analogy with (11).

Equations for iteration parameters (13a) are applicable for $i<i_0$. For $i>i_0$ they should be replaced by:

$$\omega_s = 1 - 1.5\ [1 + \frac{c}{4(I-i+1)}]R_s \text{ for } i>i_0+1, \text{ and}$$

$$\omega_s = 1 - 1.5(2 + \frac{1}{2I})R_s \text{ for } i = i_0 + 1. \qquad (13b)$$

In the definition of R_s, $\eta = \sin^2(\frac{\pi}{2J_i})$ for $i = i_0+1$, and

$$\eta = \sin^2(\frac{\pi}{2J_i}) + \sin(\frac{\pi}{2J_i})\frac{c}{2(I-i+1)} \text{ for } i > i_0 + 1.$$

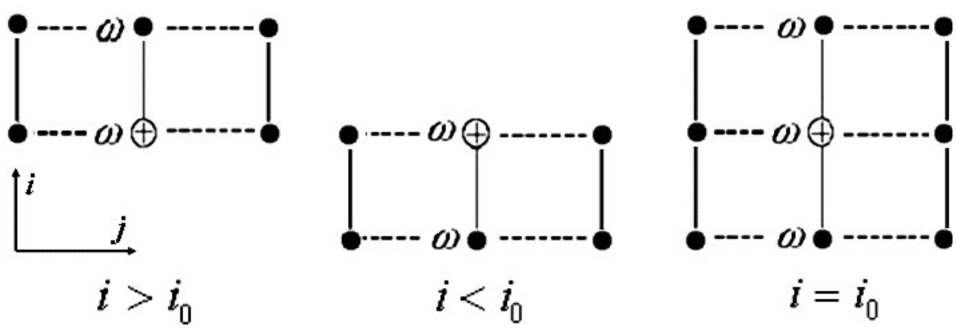

Fig. 7. Templates of the operator B.

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