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RADIATIVE EQUILIBRIUM IN THE MIDDLE ATMOSPHERE

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RESUMEN

En este trabajo se describe un estudio teórico del estado de Equilibrio Radiativo de la Atmósfera media; nuestro enfoque es básicamente analítico. Se introduce un nuevo método general de solución basado en las *funciones radiativas características* (modos naturales de decaimiento de perturbaciones de temperatura en la atmósfera). Los resultados del método son comparados con otros obtenidos por métodos puramente numéricos, mostrando excelente precisión, además de la conveniencia del uso de métodos analíticos.

ABSTRACT

We describe a theoretical study of the state of radiative equilibrium of the middle atmosphere; our approach is basically analytic. We introduce a new general method of solution based on the *radiative eigenfunctions* (natural modes of radiative temperature relaxation in the atmosphere). This method is checked against published results involving purely numerical solutions. Our results indicate good accuracy, and the convenience of workable analytic methods.

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INTRODUCTION

The temperature in the middle atmosphere is basically controlled by radiative processes. It is well known that the region comprised between 10 and 30 kilometers is in a state of approximate radiative equilibrium. The heating produced by ozone absorption of solar radiation is balanced by cooling in the infrared by carbon dioxide, water vapor and ozone.

Different computational techniques have been used to solve the problem relying basically on purely numerical methods. One approach is to compute separately the heating due to absorption of ultraviolet radiation and the transfer due to infrared radiation. The purpose of this work is to introduce an analytic approach based on the knowledge of the *radiative eigenfunctions* introduced by Gay and Thomas (1981), which allows the inclusion of different transport mechanisms and the obtainment of solutions appropriate for the whole of the atmospheric region of interest. The method is also applicable to the analytic determination of the infrared source function in the case where non-LTE conditions prevail as occurs in the upper mesosphere (Kuhn and London, 1969).

The *radiative eigenfunctions* appear in the treatment of the radiative relaxation of temperature perturbations and correspond to elementary perturbations, each one decaying at a constant rate without changing its shape.

From the mathematical point of view these *radiative eigenfunctions* are the solutions of a homogeneous Fredholm integral equation of the second kind, they form the characteristic set of the kernel. When the kernel is symmetric, the eigenfunctions are orthogonal and therefore can be used as a basis for the representation of arbitrary functions in the sense of the generalized Fourier series. This property of the *radiative eigenfunctions* is used in the solution of the radiative equilibrium problem.

In what follows we do not attempt the treatment of a realistic atmospheric case, but instead, we intend to show how the method is applied to the solution of an ideal problem which contains some of the features appearing in the atmospheric cases.

METHODOLOGY

Many problems of radiative transfer can be mathematically stated as a Fredholm integral equation of the second kind.

$$\psi(\tau) = F(\tau) + \lambda \int_0^{\tau^*} \psi(\theta)\, K(\theta, \tau)\, d\theta \qquad (1)$$

Both the problems of radiative equilibrium and the determination of the source function for non-LTE conditions fall into this category.

The solution of Eq. (1) in terms of the solutions of the homogeneous equation, when the Kernel is symmetric, $K(\theta, \tau) = K(\tau, \theta)$.

$$U_n(\tau) = \lambda_n \int_0^{\tau^*} U_n(\theta)\, K(\theta, \tau) d\theta \tag{2}$$

is a standard problem in the theory of integral equations (Krasnov *et al.*, 1970) and has a simple solution.

$$\psi(\tau) = F(\tau) + \lambda \Sigma [\int_0^{\tau^*} F(\tau') U_n(\tau') d\tau'] / [\lambda_n - \lambda]\, U_n(\tau) \tag{3}$$

The simplicity of Eq. (3) indicates that the actual problem is to find the solutions to Eq. (2). This problem has already been solved in different physical contexts (i.e. Van Trigt, 1970) with different mathematical techniques varying on the degree of difficulty and generality. Gay (1978), and later Gay and Thomas (1981) solved the problem using the kernel approximation which assures the possibility of expanding the kernel K_1, a function of the difference of the arguments, $K_1 = K_1(|\theta - \tau|)$ as a series of exponentials.

$$K_1(X) = \sum_{i=1}^{N} A_i e^{-B_i X} \tag{4}$$

This can be achieved with an extremely good accuracy (Wiscombe and Evans, 1977) and lends the method of the radiative eigenfunctions a generality lacking in the others.

The information about the absorption properties of the atmosphere is contained in the kernel. If the different kernels that result can be brought to a standard form, say the exponential-sum fit as expressed in Eq. (4) a solution based on this representation constitutes a standard solution of great generality.

THE HOMOGENEOUS INTEGRAL EQUATION

An outline of the methods of solution will be given here. The reader interested in more details is referred to the paper by Gay and Thomas.

If the kernel as expressed in Eq. (4) is substituted into Eq. (2) the solution to the resulting equation is

$$\psi_n(\tau) = \sum_{\alpha} L_\alpha e^{-k_\alpha \tau} \tag{5}$$

Where the coefficients L_α and k_α satisfy the following relationships.

$$1 = \lambda \sum_{i=1}^{N} \frac{A_i B_i}{B_i^2 - k^2} \tag{6}$$

$$\sum_\alpha \frac{L_\alpha e^{-k_\alpha \tau^*/2}}{B_i + k_\alpha} = 0 \tag{7}$$

$$\sum_\alpha \frac{L_\alpha e^{-k_\alpha \tau^*/2}}{B_i - k_\alpha} = 0 \tag{8}$$

The first relationship Eq. (6) is known as the characteristic equation. The physical interpretation of Eqs. (7) and (8) is that of no incoming perturbation intensities from outside the atmosphere; in other words, the source of the perturbation is only internal.

The simultaneous solution of Eqs. (6), (7) and (8) provides all the quantities that are necessary to define the solutions (Eq. 5). This procedure represents the first method of solution as proposed by Gay and Thomas. It turns out that solutions exist only for certain values of λ (the eigenvalue of the problem). These constitute an orthogonal set and can be used in Eq. (3).

It is necessary to stress that Eq. (5) together with Eqs. (6), (7) and (8) constitute *exact* solutions to Eq. (2) in the context of the kernel approximation.

Due to some difficulties in the numerical solution of Eqs. (6), (7) and (8), a second method was proposed based on the solutions of the simplified problem given by

$$\vartheta_n(\tau) = \nu_n \int_0^{\tau^*} \vartheta_n(\theta) A_i e^{-B_1(|\theta-\tau|)} d\theta \tag{9}$$

where A_1 and B_1 are the first coefficients, (with B_1 the largest of the B_s) in the exponential-sum of the kernel. These solutions are very simple

$$\vartheta_j^e = N_j^e \cos(k_j^e \tau) \qquad \vartheta_j^o = N_j^o \sin(k_j^o \tau) \tag{10}$$

$$\nu_j^e = ({k_j^e}^2 + B_1^2)/A_1 B_1 \qquad \nu_j^o = ({k_j^o}^2 + B_1^2)/A_1 B_1 \tag{11}$$

$$\tan(k_j^e \tau^*/2) = B_1/k_j^e \qquad \tan(k_j^o \tau^*/2) = -k_j^o/B_1 . \tag{12}$$

$$N_j^e = N_j^o = \left[\frac{\tau^*}{2} + \frac{B_1}{k_j^{(e,o)^2} + B_j^2}\right]^{-1/2} \tag{13}$$

e = even o = odd

The normalization constant N_i is chosen so that the integral of ϑ_j over all optical depths is unity.

The above solutions are orthogonal and complete, and are used to expand the more general solution Eq. (5) as

$$\psi_n(\tau) = \sum_i C_{in} \vartheta_i(\tau) \tag{14}$$

The problem of determining the coefficients C_{in} is reduced to a matrix problem which can be solved by standard subroutines as explained by Gay and Thomas.

This new representation in turn can be used to find the solution to the non-homogeneous integral equation (1) in the form given by Eq. (3).

APPLICATION OF THE METHOD

To illustrate the methods outlined previously a classical problem of radiative transfer in a plane parallel atmosphere will be solved. The problem consists in the determination of the radiative equilibrium temperature profile in an atmosphere bounded by two radiating boundaries at different temperatures T_1 and T_2. Ozisik (1973), Crosbie and Viskanta (1970) have shown that the problem can be stated in terms of a function $f(\tau)$ which satisfies the following integral equation:

$$f(\tau) = \frac{1}{2} K_2(\tau) + \frac{1}{2} \int_o^{\tau^*} f(\tau') K_1(|\tau - \tau'|) d\tau' \tag{15}$$

where $K_2(\tau)$ satisfies $dK_2(x)/dX = -K_1(X)$ and K_1 or K_2 are approximated by a series of exponentials (c.f. Eq. 4) Ozisik shows that $f(\tau)$ satisfies

$$f(\tau) = \frac{\sigma T^4(\tau) - \Pi I^-(\tau^*)}{\Pi I^+(o) - \Pi I^-(\tau^*)} \tag{16}$$

where T is the temperature and I^+, I^- are the intensities at the boundaries for a gray atmosphere. Therefore, once Eq. (15) is solved, the temperature can be obtained from Eq. (16).

In order to be able to compare with the numerical results of Crosbie and Viskanta, exponential-sum-fits were obtained for their kernel values (using Wiscombe and Evans procedures) for three different models, rectangular (gray absorption) exponential band model and Doppler lines. The results are plotted in figures 1 to 3; the continuous lines are the results of this work and the point are values from their work. It is clear from the figures that the agreement is excellent. In table 1 some values obtained with the two methods explained above are compared; the first using the "exact" solution, the other using approximate solutions. Again the agreement is quite good.

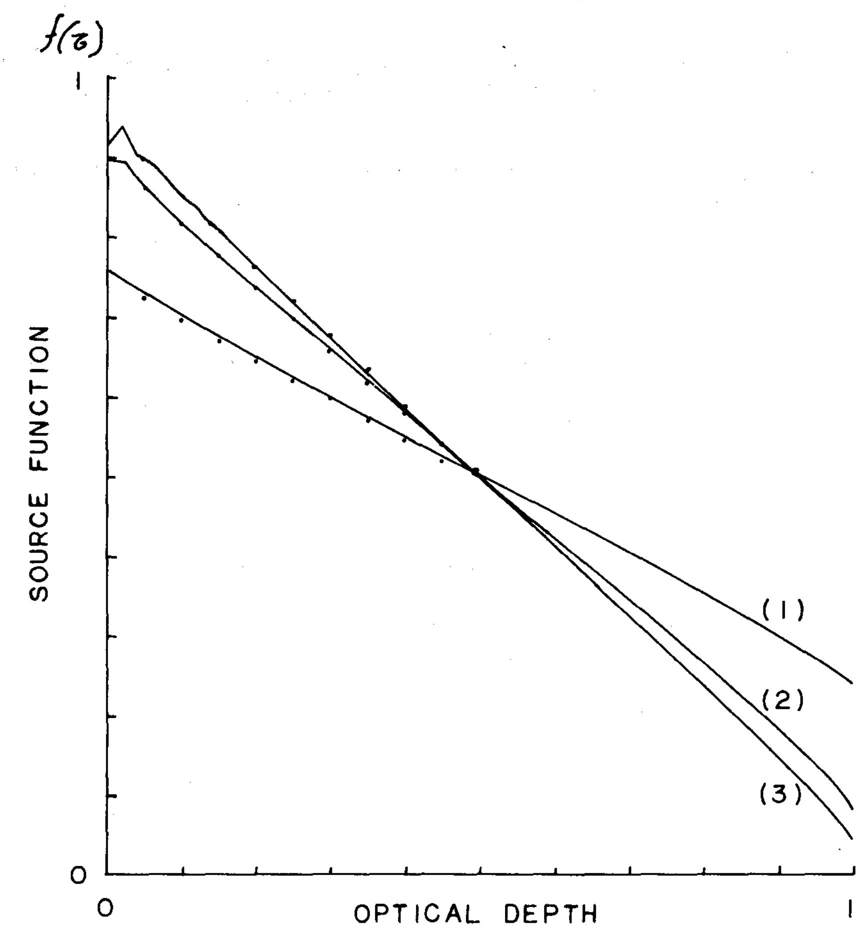

Fig. 1. Plots of the $f(\tau)$ function (source function) for a gray atmosphere and three optical depths calculated with the "approximate method". (1) $\tau^* = 1$, (2) $\tau^* = 5$ and (3) $\tau^* = 10$. The dots are values from Crosbie and Viskanta (1970).

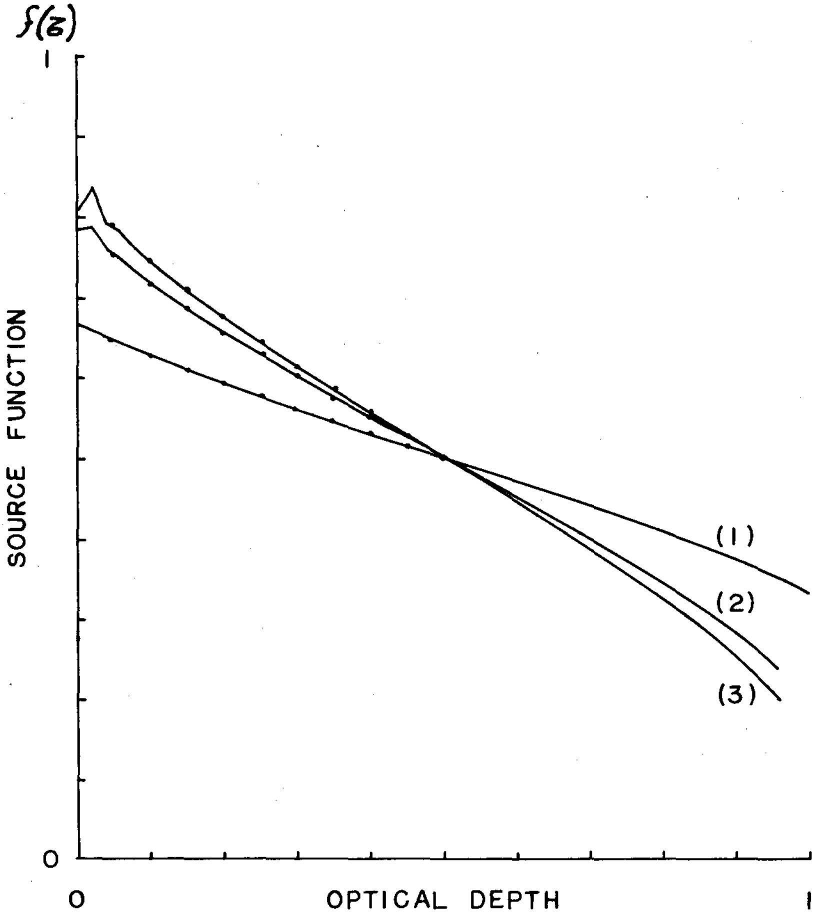

Fig. 2. Same as before, but for the Exponential Band Model.

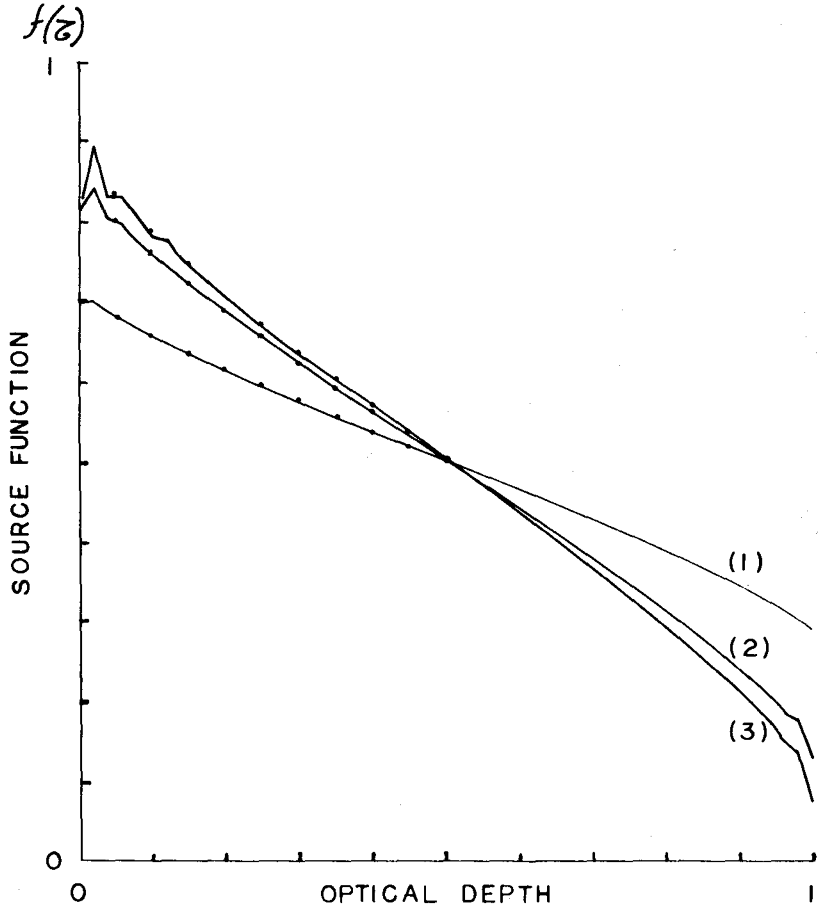

Fig. 3. Same as before but for a Doppler profile.

Table 1

Values of the source function at different positions calculated with the "exact" method and the approximate method and values from Crosbie and Viskanta.

Gray Taustar = 1			
Opt. Depth	Exact	Approx.	C. and V.
–.5	.7569	.7553	.75815
–.4	.70133	.7011	.7229
–.3	.6486	.6485	.6428
–.2	.5980	.5980	.59417
–.1	.5486	.5487	.5468
–.0	.50001	.5000	.5000

Gray Taustar = 10			
Opt. Depth	Exact	Approx.	C. and V.
–.5	1.008	.9126	.9494
–.4	.8702	.8486	.8513
–.3	.7829	.7610	.7629
–.2	.6963	.6735	.6752
–.1	.6047	.5860	.5876
–.0	.5113	.4985	.5000

Exponential Taustar = 1			
Opt. Depth	Exact	Approx.	C. and V.
–.5	.6655	.6663	.66855
–.4	.6288	.6290	.6251
–.3	.5965	.5939	.59108
–.2	.5655	.5614	.5596
–.1	.5333	.5303	.5297
–.0	.50001	.4999	.5000

Exponential Taustar = 10			
Opt. Depth	Exact	Approx.	C. and V.
	.9035	.8072	.8538
	.7646	.7451	.7477
	.6898	.6743	.6959
	.6233	.6123	.6136
	.5605	.5546	.5559
	.50008	.4988	.5000

DISCUSSION

The results presented above show an alternative analytic approach for the treatment of steady state atmospheric problems; many details remain to be included, to make possible a "realistic" application of the method. The inhomogeneity of the atmosphere can be taken into account with a suitable definition of the optical depth.

Some work in progress indicates that the methodology here presented can be extended to the solution of equations with coefficients depending on position. As discussed by Gay and Thomas, in an inhomogeneous atmosphere the kernel is not symmetric but a change in the dependent variable turns the problem into one susceptible of being treated with the methods presented here.

Another problem of interest for the middle atmosphere is the coupling of radiation and convection leading to a state of radiative-convective equilibrium. This problem, being non-linear, presents special problems. A possible approach using the *radiative eigenfunction* and perturbation analysis is indicated next.

It can be shown that a very simplified statement of a radiative-convective problem can be mathematically expressed as

$$S(\tau) = A\frac{\partial^2 T}{\partial \tau^2} + B\left\{\frac{1}{2}\int_0^{\tau^*} S(\tau')K_1(|\tau' - \tau|)d\tau'\right\} + S^*(\tau) \tag{17}$$

where the S* contains the boundary terms.

The first term refers to convection (with a constant eddy coefficient) and the second to radiation. If the first or second term is small, a solution can be found.

Assuming a small convection, a solution is proposed as an expansion in powers of A, the small parameter. Then the temperature and the source function can be written as:

$$T = T_0 + AT_1 + A^2 T_2 + \ldots \tag{18}$$

and

$$S = S_0 + \frac{\partial S}{\partial T}\Big|_{T_0} AT_1 + \ldots \tag{19}$$

substituting these expansions in Eq. (17) and comparing the coefficients of equal powers of A yields the following equations

$$S_0(\tau) = \frac{B}{2}\int_0^{\tau^*} S_0(\tau')K_1(|\tau' - \tau|)d\tau' + S^*(\tau) \tag{20}$$

and

$$\frac{\partial S}{\partial T}\Big|_{T_0} T_1 = \frac{\partial^2 T_0}{\partial \tau^2} + \frac{B}{2}\int_0^{\tau^*} \frac{\partial S}{\partial T}\Big|_{T_0} T_1 K_1(|\tau' - \tau|)\,d\tau' \tag{21}$$

Equation (20) corresponds to the condition of radiative equilibrium, and its solution has already been discussed. The solution of Eq. (20) provides T_0 which in turn is substituted in the derivative term of Eq. (21) yielding again an inhomogeneous integral equation, that can be readily solved, for $\frac{\partial S}{\partial T}|_{T_0} T_1$ and then for T_1, this completes the solution up to the linear terms. It is clear that the procedure could be continued to higher order terms until achieving the desired accuracy.

The case just discussed corresponds to the problem of radiative equilibrium perturbed by convection. A similar procedure can be followed in the opposite case, that is, convective equilibrium perturbed by radiation.

CONCLUSION

It is believed that the method here introduced provides a rather simple way of treating atmospheric problems, in which radiative transfer intervenes. Linear problems admit direct solutions; non-linear problems susceptible of linearization can also be solved. These need more careful analysis.

The method also provides new insights into the theory of radiative transfer. The *radiative eigenfunctions* play a preponderant role and appear naturally in many different problems. They are the natural modes of a radiant atmosphere; any temperature profile (source function) may be obtained by a suitable combination of them.

The good results shown in the examples besides the relative simplicity of the method as compared to others indicate that more studies should be conducted to refine it. This is already being done by the authors.

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