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A balanced and absolutely stable numerical thermodynamic model for closed and open oceanic basins

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RESUMEN

Por medio de condiciones especiales de frontera, el problema bien establecido se formula para el modelo oceánico termodinámico de Adem en una región de océano abierta, cuando existe un flujo anómalo de calor a través de las fronteras laterales. Se muestran la unicidad y estabilidad de las soluciones del modelo. Se estima la velocidad de disipación de las anomalías de temperatura en presencia de la difusión y ausencia del forzamiento.

Se muestra que el operador del modelo es positivo definido, positivo semidefinido o antisimétrico, dependiendo del tipo de condiciones de frontera y de la difusión. El método de separación se aplica para construir un esquema implícito en diferencias finitas con aproximación de segundo orden, el cual es económico, balanceado e incondicionalmente estable. Cada uno de los problemas separados es de dimensión 1, y se resuelve fácilmente por el método de factorización. Se justifica la aplicación del método de separación. El algoritmo numérico se puede generalizar con facilidad para el modelo de 3-dimensiones.

PALABRAS CLAVE: Anomalías de temperatura superficiales, modelo termodinámico, condiciones de frontera, estabilidad.

ABSTRACT

By setting special boundary conditions the well-posed problem is formulated for the Adem thermodynamic model in an open oceanic basin when there is an anomalous heat flow across the lateral boundaries. Uniqueness and stability of the model solutions are shown. Estimates of the rate of dissipation of the temperature anomalies in the presence of diffusion and the absence of forcing are provided.

The model operator is positive definite, positive semidefinite or skew-symmetric depending on the boundary conditions type and the diffusion. The splitting method is applied to construct an implicit 2nd order finite-difference scheme that is economical, balanced, and unconditionally stable. Each of the split problems is one-dimensional and can easily be solved by factorization. The numerical algorithm can readily be generalized to three dimensions.

KEY WORDS: Sea surface temperature anomaly, thermodynamic model, open boundary conditions, balanced stable scheme.

1. INTRODUCTION

The thermodynamic model developed by Adem (1964, 1971, 1975, 1991), Adem and Mendoza (1988) and Adem *et al.* (1991, 1994) for the upper mixed level of the ocean may be used for calculating monthly mean sea surface temperature (SST) anomalies. In the case of a closed oceanic basin when there is no heat flow across the boundary, the model differential operator is positive definite, positive semidefinite or skew-symmetric depending on boundary conditions and the presence or absence of diffusion. This allows us to formulate the well-posed problem in the sense of Hadamard (1923) where solutions are unique and stable. A model of this type is briefly described in sections 2-4.

Many references deal with the problem of boundary conditions for an open basin (see e. g. Poinsot and Lele, 1992, and the appended references). In section 5, appropriate boundary conditions are formulated for the Adem thermodynamic model in an open oceanic basin with heat flow across the boundaries. It is shown in section 6 that with these boundary conditions, the open basin thermodynamic model is also a well-posed problem in the sense of Hadamard, and that the decay of temperature anomalies for different scales is determined by the eigenvalue of the model operator.

In section 7, the 2-D differential operator of the model is split into the sum of two 1-D operators. If the original operator is positive definite (positive semidefinite or skew-symmetric), then each of the split operators has the same property. The finite-difference approximations to the original and each of the split operators preserve this property, justifying the application of the splitting method (Yanenko, 1971) to constructing a balanced and unconditionally stable implicit scheme that has two conservation laws in the absence of dissipation and forcing.

The numerical algorithm can readily be generalized to three dimensions. The solution of the original 3-D problem is reduced to solving a few simple 1-D problems.

Difference split operators and boundary conditions of the second-order approximation in space and time are constructed in section 8. A balanced implicit scheme absolutely stable to initial perturbations is given in section 9, and final conclusions are made in section 10.

2. FORMULATION OF THE MODEL IN A CLOSED OCEANIC BASIN

Let Ω be a two-dimensional closed ocean basin with a lateral ocean boundary S. Consider in Ω and for a time interval $(0, \bar{t})$ the Adem (1971) ocean thermodynamic model governed by the equation

$$\frac{\partial T}{\partial t}+\mathbf{U}\cdot\nabla T-\mu\nabla^2 T=F \quad (1)$$

where $T(\mathbf{r},t)$ is the climatic ocean temperature anomaly; $\mathbf{U}(\mathbf{r},t)$ is the climatic seasonal current velocity vector; $F(\mathbf{r},t)$ is the heat forcing anomaly that includes evaporation, radiation, sensible heat transfer, and advection of heat by anomalous current velocity; $\mathbf{r}\equiv(\lambda,\theta)$ is a point in Ω; μ is the constant horizontal diffusion coefficient ($\mu>0$), and ∇ is the two-dimensional (spherical) gradient. The terms $\mathbf{U}\cdot\nabla T$ and $\mu\nabla^2 T$ in Eq.(1) describe the change of temperature anomaly under the effects of advection and turbulent diffusion, respectively.

It is assumed that the climatic velocity vector $\mathbf{U}$ satisfies the continuity equation

$$\operatorname{div}\,\mathbf{U}\equiv\frac{1}{a\sin\theta}\left\{\frac{\partial u}{\partial\lambda}+\frac{\partial}{\partial\theta}(v\sin\theta)\right\}=0 \quad (2)$$

where a is the earth's radius, λ is the longitude, θ is the colatitude, and u and v are the components of $\mathbf{U}$ in spherical coordinates. Since the oceanic basin Ω does not contain open (liquid) segments, the normal component U_n of the current velocity $\mathbf{U}$ is zero at $\mathbf{S}$:

$$U_n\equiv\mathbf{U}\cdot\mathbf{n}=0 \quad \text{at } \mathbf{S} \quad . \quad (3)$$

Here $\mathbf{n}$ is the outward normal to $\mathbf{S}$. Equation (1) is to be solved with the initial condition

$$T(\mathbf{r},0)=T^0(\mathbf{r}) \quad \text{in } \Omega \quad (4)$$

and the boundary condition

$$\mu\frac{\partial T}{\partial n}=0 \quad \text{at } \mathbf{S} \quad (5)$$

where $\partial/\partial n$ is the derivative in the direction of the outward normal. The conditions (3) and (5) mean that there is no anomalous heat flow across the boundary $\mathbf{S}$.

3. UNIQUENESS AND STABILITY OF THE CLOSED BASIN MODEL SOLUTIONS

Let

$$\|T\|=\left(\int_\Omega|T(\mathbf{r})|^2\,dr\right)^{1/2} \quad (6)$$

be a norm of the function $T(\mathbf{r})$ and let $\mathbf{H}$ be a Hilbert space of all complex functions whose norm (6) is finite. We define the inner product

$$<T,g>=\int_\Omega T(\mathbf{r})\overline{g(\mathbf{r})}\,\mathrm{d}\mathbf{r} \quad (7)$$

for functions $T(\mathbf{r})$ and $g(\mathbf{r})$ of $\mathbf{H}$. Here $\overline{g(\mathbf{r})}$ denotes the complex conjugate of $g(\mathbf{r})$. Because of (2), the differential operator of the problem (1)-(5) can be written as

$$AT=\operatorname{div}(\mathbf{U}T)-\mu\nabla^2 T \quad (8)$$

which is defined for all sufficiently smooth functions $T(\mathbf{r})$ that satisfy the boundary condition (5). We now show that A is positive semidefinite, that is,

$$<AT,T>\geq 0 \quad (9)$$

for each function $T(\mathbf{r})$ of its domain (this property will be denoted by $A\geq 0$). Integrating the inner product $<AT,T>$ by parts and using Green's formula and conditions (3) and (5), we obtain

$$<AT,T>\equiv\int_\Omega TAT\mathrm{d}\mathbf{r}=\mu\int_\Omega|\nabla T|^2\mathrm{d}\mathbf{r}\geq 0 \quad . \quad (10)$$

In the nondissipative case ($\mu\equiv 0$), the operator A is skew-symmetric:

$$<AT,T>\equiv 0\,. \quad (11)$$

Also, if $g(\mathrm{x})\equiv 1$ in Ω then

$$<AT,g>\equiv 0\,. \quad (12)$$

Thus, integrating (1) over Ω we obtain the heat balance equation

$$\frac{\partial}{\partial t}\int_\Omega T\ \mathrm{d}\mathbf{r}=\int_\Omega F\ \mathrm{d}\mathbf{r}\,. \quad (13)$$

Hence the average temperature anomaly

$$\overline{T}=(\text{mes }\Omega)^{-1}\int_\Omega T\ (\mathbf{r})\ \mathrm{d}\mathbf{r}\ , \quad (14)$$

where mes Ω is the area of Ω, will change if and only if the average heat forcing anomaly

$$\overline{F}=(\text{mes }\Omega)^{-1}\int_\Omega F\ (\mathbf{r})\ \mathrm{d}\mathbf{r} \quad (15)$$

is nonzero. Taking the inner product (7) of Eq.(1) with the solution T and using (9) we find

$$\|T(\mathbf{r},t)\|^2\leq\left\|T^0(\mathbf{r})\right\|^2+2\int_0^t<F(\mathbf{r},t),T(\mathbf{r},t)>\mathrm{d}t \quad (16)$$

or

$$\|T(\mathbf{r},t)\|\ \leq\ \left\|T^0(\mathbf{r})\right\| \quad (17)$$

when $F(\mathbf{r},t)\equiv 0$. It follows from (16) and (17) that each solution to the linear problem (1)-(5) is unique and stable to initial perturbations. Hence the problem (1)-(5) is well-posed in the sense of Hadamard (1923).

4. RATE OF DECAY OF THE SST ANOMALIES

It follows from (13) that the average temperature anomaly (14) is constant if the average heat forcing anomaly (15) is zero. Let now the forcing anomaly F of Eq.(1) be identically zero. By (9)-(11), the model operator (8) is skew-symmetric if μ=0, or positive definite if μ>0. Therefore the norm $\|T(\mathbf{r},t)\|$ will be conserved in the first case, and decreased in the second one. We now estimate the rate of decay. For this purpose let us consider the spectral problem

$$A\, W_m(\mathbf{r}) = \omega_m\, W_m(\mathbf{r}) \tag{18}$$

$$\mu \frac{\partial}{\partial n} W_m(\mathbf{r}) = 0 \quad \text{at } \mathbf{S} \tag{19}$$

for the operator (8). According to (9), all eigenvalues ω_m are positive except for the zero eigenvalue ω_1 which corresponds to a constant eigenfunction $W_1(\mathbf{r})$. We take

$$W_1(\mathbf{r}) = (\text{mes}\,\Omega)^{-1/2} \tag{20}$$

to satisfy $\|W_1(\mathbf{r})\|$=1. Since A is non-symmetric, the eigenfunctions are not orthogonal in the sense of the inner product (7). Thus we also consider the eigenvalue problem

$$A^* G_k(\mathbf{r}) = \overline{\omega}_k G_k(\mathbf{r}) \tag{21}$$

for the adjoint operator

$$A^* G = -\text{div}(\mathbf{U}G) - \mu\nabla^2 G \tag{22}$$

with the same boundary condition (19) for G_k at $\mathbf{S}$. Here $\overline{\omega}_k$ is the complex conjugate of ω_k. The eigenfunctions $W_m(\mathbf{r})$ and $G_k(\mathbf{r})$ make the biorthogonal basis in the space $\mathbf{H}$:

$$\langle W_m, G_k \rangle = \delta_{mk} \tag{23}$$

where δ_{mk} is the Kronecker symbol. Evidently, $G_1 = W_1$. The functions $W_k(\mathbf{r})$ and $G_k(\mathbf{r})$ can be used to characterize a spatial scale of the temperature anomalies in the oceanic domain Ω. Note that their scale decreases as k increases. Let us expand the solution $T(\mathbf{r},t)$ as

$$T(\mathbf{r},t) = \sum_{m=1}^{\infty} T_m(t) W_m(\mathbf{r}) \ , \tag{24}$$

where the coefficient

$$T_m(t) = \langle T(\mathbf{r},t), G_m(\mathbf{r}) \rangle \tag{25}$$

represents the amplitude of a temperature anomaly that coincides with $W_m(\mathbf{r})$. Thus $T_m(t)$ characterizes the contribution of the temperature anomaly of some particular scale m to the total temperature anomaly $T(\mathbf{r},t)$.

The inner product of the homogeneous Eq. (1) with $G_k(\mathbf{r})$ gives

$$\frac{\partial}{\partial t} T_k + \omega_k T_k = 0 \tag{26}$$

or

$$T_k(t) = T_k^0 \exp(-\omega_k t) \ , \tag{27}$$

where T_k^0 is the corresponding coefficient of the initial temperature anomaly (4). The expression (27) tends to zero for $k \geq 2$, and the rate of decay of the SST anomaly for a spatial structure and scale of a function W_k is determined by the eigenvalue ω_k. The amplitude T_1 will be constant according to (13) and the assumption $F = 0$:

$$T_1 = \frac{1}{\sqrt{\text{mes}\ \Omega}} \int_\Omega T(\mathbf{r})d\mathbf{r} = \frac{1}{\sqrt{\text{mes}\ \Omega}} \int_\Omega T^0(\mathbf{r})d\mathbf{r} \ . \tag{28}$$

As a result, the temperature anomaly will eventually tend to the mean value (14) of the initial temperature anomaly. In particular, if T^0= 0 the solution to the problem (1)- (5) will tend to zero with time.

The above eigenvalue method can be applied to an oceanic domain Ω of arbitrary form and size. It enables us to choose the horizontal diffusion coefficient μ in such a way that the rate of dissipation of the SST anomalies coincides with observed value in Ω.

We now consider an example of the model with turbulent diffusion alone. Let (x,y) be the coordinates of a point $\mathbf{r}$ in a rectangular ocean basin Ω with sides $\mathbf{X}$ and $\mathbf{Y}$. It is assumed that $\mathbf{U}\equiv 0$ in Ω. Then the operator $AT = -\mu\nabla^2 T$ of Eq.(1) is symmetric, and the spectral problem (18), (19) yields a complete set of orthogonal eigenfunctions

$$W_{km}(x,y) = \cos\left(k\frac{\pi}{\mathbf{X}}x\right)\cos\left(m\frac{\pi}{\mathbf{Y}}y\right)$$

corresponding to the different non-negative eigenvalues

$$\omega_{km} = \mu\pi^2\left[\left(\frac{k}{\mathbf{X}}\right)^2 + \left(\frac{m}{\mathbf{Y}}\right)^2\right] \quad (k,\ m = 0,1,2,\ldots) \ .$$

Then, by (27), the amplitude of an initial temperature anomaly in the form of $W_{km}(x,y)$ will decay as

$$T_{km}(t) = T_{km}^0 \exp(-\omega_{km} t) \ . \tag{29}$$

A simple case of (29) when $\mathbf{X}$=$\mathbf{Y}$ was considered by Adem (1971, Figures 4 and 6).

5. FORMULATION OF THE MODEL IN AN OPEN OCEANIC BASIN

Now consider the case of an open oceanic basin Ω when anomalous heat flow is possible across the liquid parts of the boundary $\mathbf{S}$. As above, the conditions (3) and (5) will be used at the boundary coinciding with the coast line. Physically and mathematically appropriate conditions at the liquid boundary have been discussed very widely (see, e.g., Vichnevetsky and Bowles, 1982; Poinsot and Lele, 1992, and references in these papers). Here the approach suggested by Marchuk (1986) for the 2-D problem, and by Skiba (1993) for the 3-D problem, will be used.

Let us divide the lateral boundary **S** into two parts: the "outflow" part $\mathbf{S}^+$ where the velocity vector **U** is outwardly directed ($U_n \geq 0$), and the "inflow" part $\mathbf{S}^-$ where the velocity vector **U** is inwardly directed ($U_n \leq 0$). On Figure 1 taken from Adem *et al.* (1994), the Gulf of Mexico is an example of such an open basin.

Equation (1) is now solved in the time interval $(0, \bar{t})$ and over the domain Ω with initial condition (4) and boundary conditions

$$\mu \frac{\partial T}{\partial n} - U_n T = 0 \quad \text{at } \mathbf{S}^- , \tag{30}$$

$$\mu \frac{\partial T}{\partial n} = 0 \quad \text{at } \mathbf{S}^+ . \tag{31}$$

Condition (30) means that the combined diffusive plus advective anomalous heat inflow is zero at $\mathbf{S}^-$ (nonzero climatic heat flows at $\mathbf{S}^-$ are however possible; see Figure 1). If (30) is violated, and there is a SST anomaly source outside of Ω that generates a heat flow at $\mathbf{S}^-$, then by widening the basin Ω this source can be included in Ω. On the other hand, (30) can be replaced by

$$\mu \frac{\partial T}{\partial n} - U_n T = q \quad \text{at } \mathbf{S}^- , \tag{32}$$

if the anomalous heat flow $q(\mathbf{r}, t)$ is known at $\mathbf{S}^-$ from observations. Condition (31) means that at the boundary $\mathbf{S}^+$ the anomalous heat flow induced by diffusion is negligible as compared with that caused by advection with the current velocity vector **U**.

Where **S** coincides with the coast line Eqs.(30) and (31) automatically satisfy condition (5) due to (3). Thus when (3) is imposed at the solid part of **S**, conditions (30) and (31) can be used at the entire boundary **S** without separating into solid and liquid segments.

Eqs. (30) and (31) are well known boundary conditions of the third kind and the second kind, respectively (Ladyzhenskaya, 1973). In the limiting no-diffusive case (μ=0), (30) is reduced to the reasonable condition T=0 at the inflow part of the boundary, while condition (31) vanishes as it should, since for pure advection, no condition is required at the outflow boundary, where the solution is defined by the method of characteristics (Godunov, 1971).

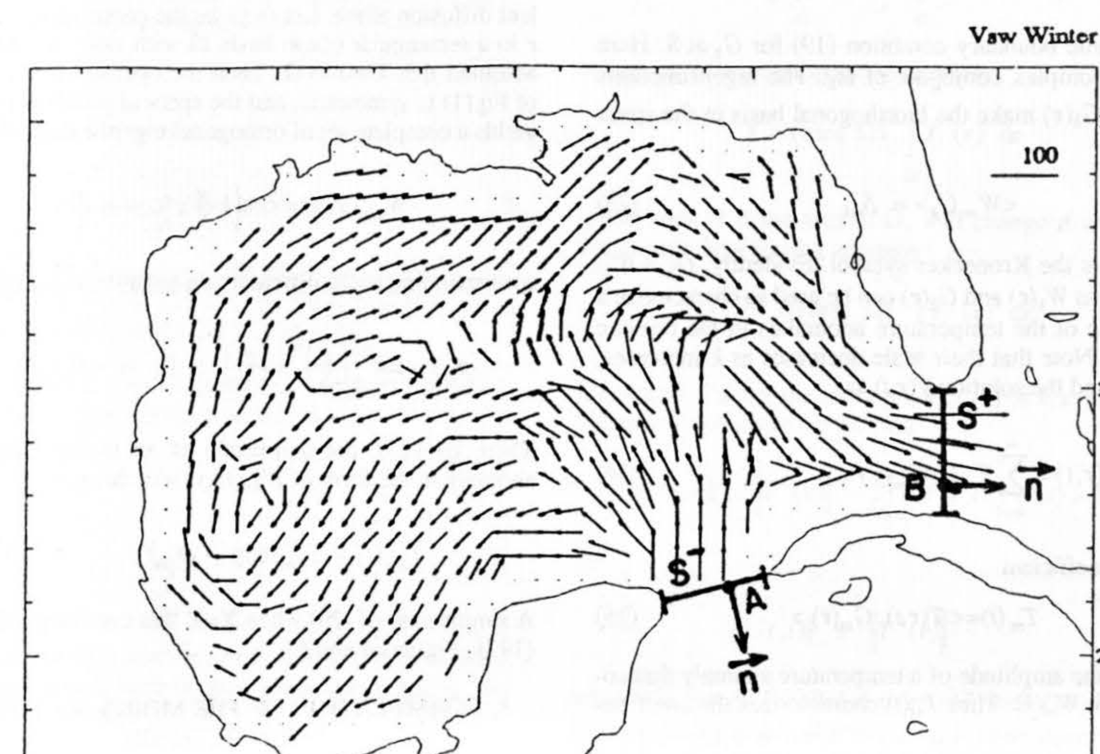

Fig. 1. The open oceanic domain **D** for the Gulf of Mexico. Points A and B belong to the boundary parts $\mathbf{S}^-$ and $\mathbf{S}^+$ respectively: $U_n(A) < 0$ and $U_n(B) > 0$ where U_n is a projection of the current velocity vector **U** on the outward unit normal **n** to the boundary.

6. UNIQUENESS AND STABILITY OF THE OPEN BASIN SOLUTIONS

We show now that the operator A of the problem (1), (4), (30), (31) is positive semidefinite: $A \geq 0$. Integrating the inner product $<AT,T>$ by parts and using Green's formula we obtain

$$< AT,T >= \mu \int_{\Omega} |\nabla T|^2 \mathrm{dr} + \frac{1}{2}\left\{ \int_{S^+} U_n T^2 \mathrm{dS} - \int_{S^-} U_n T^2 \mathrm{dS} \right\} \geq 0 \qquad (33)$$

where dS is a linear element of the boundary S. The last two integrals in (33) are non-negative since $U_n \geq 0$ at S^+, and $U_n \leq 0$ at S^-. Thus the operator A is non-negative. In particular, A is positive definite ($A > 0$) when U_n is not identically zero at S and/or $\mu>0$. Then all eigenvalues ω_k of the spectral problem

$$\begin{aligned} &A\, W_k(\mathbf{r}) = \omega_k\, W_k(\mathbf{r}) \\ &\mu \frac{\partial}{\partial n} W_k - U_n W_k = 0 \quad \text{at } S^- , \\ &\mu \frac{\partial}{\partial n} W_k = 0 \quad \text{at } S^+ \end{aligned} \qquad (34)$$

are positive. As we saw in section 4, if $F \equiv 0$ the amplitude T_k^0 of an initial temperature anomaly $W_k(\mathbf{r})$ decays as the rate determined by the eigenvalue ω_k (see (27)). Note that A is skew-symmetric only if $\mu \equiv 0$ (the no-dissipative case) and if in addition U_n is identically zero at S.

The inequalities (16) and (17) are again satisfied because of (33). Then each solution of the problem (1), (4), (30), (31) is unique and stable under initial perturbations, and the problem of the open basin is also well-posed in the sense of Hadamard.

Integrating (1) over Ω we obtain the balance equation

$$\frac{\partial}{\partial t} \int_{\Omega} T \, \mathrm{dr} = \int_{\Omega} F \, \mathrm{dr} - \int_{S^+} U_n T \mathrm{dS} + \int_{S^-} q \, \mathrm{dS} \qquad (35)$$

where boundary condition (32) is taken into account. For boundary condition (30), the last integral in the formula (35) is zero. Unlike what happens in the closed basin model, the average temperature anomaly (14) can be changed both by an anomalous heat forcing $F(\mathbf{r},t)$, and by an anomalous heat flow across the boundary S.

7. SPLITTING METHOD

Equation (1) can be written

$$\frac{\partial T}{\partial t} + (A_1 + A_2) T = F \qquad (36)$$

where

$$A_1 T = \frac{1}{2a \sin\theta}\left[\frac{\partial}{\partial \lambda}(uT) + u\frac{\partial T}{\partial \lambda}\right] - \frac{\mu}{a^2 \sin^2\theta}\frac{\partial^2}{\partial \lambda^2} T \qquad (37)$$

$$\begin{aligned} A_2 T = &\frac{1}{2a \sin\theta}\left[\frac{\partial}{\partial \theta}(vT\sin\theta) + v\sin\theta\frac{\partial T}{\partial \theta}\right] \\ &- \frac{\mu}{a^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) . \end{aligned} \qquad (38)$$

Using the continuity equation (2) it is easy to see that the sum $A_1 + A_2$ coincides with the operator A (see (8)).

We now prove that the operators (37) and (38) are positive semidefinite: $A_i \geq 0$ (i = 1, 2). Let us calculate the inner product $<A_1T,T>$. Integrating by parts and using Green's formula with the boundary conditions (30) and (31) we obtain

$$\begin{aligned} < A_1 T, T > &\equiv \int_{\Omega} T\, A_1 T \, \mathrm{dr} = \mu \int_{\Omega} \frac{1}{a^2 \sin^2\theta}\left(\frac{\partial T}{\partial \lambda}\right)^2 \mathrm{dr} \\ &+ \int_{S}\left(\frac{1}{2}U_n T - \mu\frac{\partial T}{\partial n}\right) T \mathrm{dS} = \mu \int_{\Omega} \frac{1}{a^2 \sin^2\theta}\left(\frac{\partial T}{\partial \lambda}\right)^2 \mathrm{dr} \\ &+ \frac{1}{2}\left\{\int_{S^+} U_n T^2 \mathrm{dS} - \int_{S^-} U_n T^2 \mathrm{dS}\right\} \geq 0 , \end{aligned} \qquad (39)$$

taking into account that $U_n \leq 0$ at S^-, and $U_n \geq 0$ at S^+. The positive semidefiniteness of operator (38) can be shown in the same way. Since $A_1 \geq 0$ and $A_2 \geq 0$, the splitting method is justified (Douglas and Rachford, 1956; Yanenko, 1971; Marchuk, 1975). It can be applied for solving (36) within each time interval (t, $t+\tau$) of small length τ. The method consists of two steps:

(1) the equation

$$\frac{\partial}{\partial t} T_1 + A_1 T_1 = 0 \qquad (40)$$

is solved in (t, $t+\tau$) with the initial condition $T_1(t) = T(t)$, where $T(t)$ is the solution of Eq.(36).

(2) the equation

$$\frac{\partial}{\partial t} T_2 + A_2 T_2 = F \qquad (41)$$

is solved in (t, $t+\tau$) with the initial condition $T_2(t) = T_1(t+\tau)$.

As a result, the solution $T_2(t+\tau)$ of the split problem (41) will approximate the solution $T(t+\tau)$ of the original problem (36) at time $t+\tau$.

According to Douglas and Rachford (1956), the algorithm (40), (41) has the second-order approximation in τ only if the operators A_i are time-independent or commute between each other, i.e., $A_1 A_2 = A_2 A_1$. To achieve a *2nd* order approximation in τ for the time-dependent non-commuting operators (37) and (38), we use the symmetric version of the splitting method (Marchuk, 1975; Marchuk *et al.*, 1975; Marchuk and Skiba, 1978) within each double time interval $(t-\tau, t+\tau)$. This consists of the three steps:

(1) equation (40) is solved in the interval $(t-\tau, t)$ with the initial condition $T_1(t-\tau) = T(t-\tau)$;

(2) equation (41) is solved in $(t-\tau, t+\tau)$ with the initial condition $T_2(t-\tau) = T_1(t)$;

(3) equation (40) is now solved in the interval $(t, t+\tau)$ with the initial condition $T_1(t) = T_2(t+\tau)$.

As a result, $T(t+\tau) \approx T_1(t+\tau)$.

8. FINITE-DIFFERENCE APPROXIMATION

Previous authors have constructed numerical schemes for the transport equation with some useful properties (see, for example, Forester, 1977; Smolarkiewicz, 1991; Williamson, 1992). Our goal here is to obtain the balanced and absolutely stable *2nd* order scheme that preserves the finite-difference analogies of (14) and $\|T(\mathbf{r},t)\|$ when the operator A is skew-symmetric (see section 6). These properties are especially important in long-term calculations with climatic models.

Note that the boundary **S** consists only of segments parallel to the lines λ = Const or θ = Const. Thus for every point of **S** the normal component U_n of the current velocity **U** coincides with either $\pm u$ or $\pm v$, where u, v are the components of **U** in the direction of the axes λ and θ.

Consider evenly spaced grids on a sphere with distances $\Delta\lambda$ and $\Delta\theta$ between grid points. The net functions $T_{ij} \equiv T(\lambda_i, \theta_j)$, $u_{ij} \equiv u(\lambda_{i-1/2}, \theta_j)$ and $v_{ij} \equiv v(\lambda_i, \theta_{j-1/2})$ are defined on different grids (Figure 2). The boundary line **S** of the grid domain passes through nodes of the functions u_{ij} or v_{ij} (Figure 3). We take

$$\frac{1}{a\sin\theta_j}\left[\frac{u_{i+1,j}-u_{ij}}{\Delta\lambda}+\frac{v_{i,j+1}\sin\theta_+ - v_{ij}\sin\theta_-}{\Delta\theta}\right]=0 \quad (42)$$

as the difference form of the continuity equation (2). The notation $\sin\theta_+ \equiv \sin\theta_{j+1/2}$, $\sin\theta_- \equiv \sin\theta_{j-1/2}$ is used in (42) and thereafter. In order to approximate the operators (37), (38) we note that they can be written as

$$A_1T=\frac{1}{a\sin\theta}\left[\frac{1}{2}T\frac{\partial u}{\partial\lambda}+u\frac{\partial T}{\partial\lambda}\right]-\frac{\mu}{a^2\sin^2\theta}\frac{\partial^2}{\partial\lambda^2}\mathrm{T} \quad (43)$$

$$A_2T=\frac{1}{a\sin\theta}\left[\frac{1}{2}T\frac{\partial}{\partial\theta}(v\sin\theta)+v\sin\theta\frac{\partial T}{\partial\theta}\right]$$

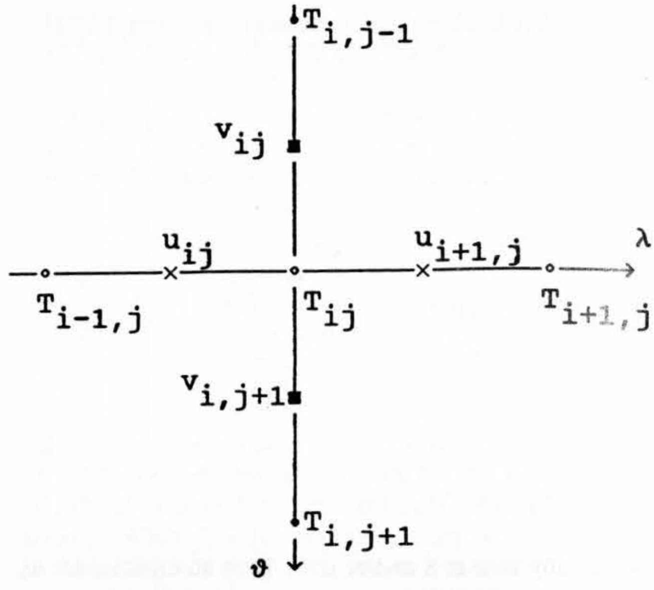

Fig. 2. Location of the grid functions near a node (λ_i, θ_j).

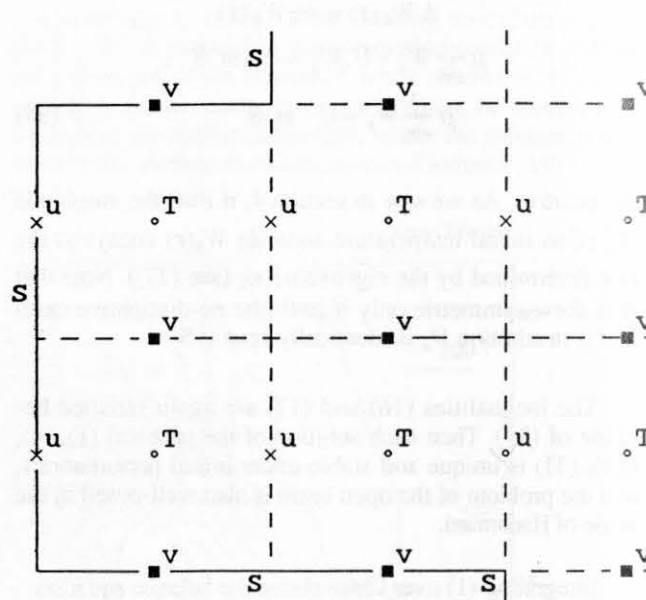

Fig. 3. The boundary line S of the grid domain passing through the nodes of the functions u_{ij} and v_{ij}.

$$-\frac{\mu}{a^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right). \quad (44)$$

Since the term

$$\frac{1}{2\Delta\lambda}\left\{u_{i+1,j}T_{i+1,j}-u_{ij}T_{i-1,j}\right\} \quad (45)$$

approximates the sum $\frac{1}{2}T\frac{\partial u}{\partial\lambda}+u\frac{\partial T}{\partial\lambda}$ at the point (λ_i, θ_j) to the *2nd* order, the operators A_m are approximated by the matrices A_m^h (Skiba 1993):

$$(A_1^h T)_{ij} = \frac{1}{2a\Delta\lambda\sin\theta_j}\left\{u_{i+1,j}T_{i+1,j} - u_{ij}T_{i-1,j}\right\}$$

$$-\frac{\mu}{a^2\Delta\lambda^2\sin^2\theta_j}\left\{T_{i+1,j} - 2T_{ij} + T_{i-1,j}\right\} , \qquad (46)$$

$$(A_2^h T)_{ij} = \frac{v_{i,j+1}\sin\theta_+ T_{i,j+1} - v_{ij}\sin\theta_- T_{i,j-1}}{2a\Delta\theta\sin\theta_j}$$

$$-\frac{\mu\left\{\sin\theta_+(T_{i,j+1} - T_{ij}) - \sin\theta_-(T_{i,j} - T_{ij-1})\right\}}{a^2\Delta\theta^2\sin\theta_j} . \qquad (47)$$

We now approximate the boundary conditions (Skiba, 1993). Consider, for example, a line along the λ-axis for a fixed θ_j. Let us assume that (λ_i, θ_j) are internal nodes of the grid domain for i=1,..., I-1, and let $\mathbf{L} \equiv (\lambda_{1/2}, \theta_j)$ and $\mathbf{R} \equiv (\lambda_{I-1/2}, \theta_j)$ be the left and right boundary points on this line (Figure 4). Then the boundary conditions (30) and (31) are approximated as follows:

(1) If $u_{1j} \equiv u(\mathbf{L}) \geq 0$ then $U_n = -u_{1j} \leq 0$ and the point **L** belongs to the boundary $\mathbf{S}^-$. Furthermore, since

$$\frac{\partial T}{\partial n}(\mathbf{L}) \cong \frac{1}{a\Delta\lambda\sin\theta_j}(T_{oj} - T_{1j}) \qquad (48)$$

the boundary condition (30) is approximated by

$$\frac{\mu}{a\Delta\lambda\sin\theta_j}(T_{oj} - T_{1j}) + \frac{1}{2}u_{1j}(T_{oj} + T_{1j}) = 0 . \qquad (49)$$

(2) If $u_{1j} \equiv u(\mathbf{L}) \leq 0$ then $U_n = -u_{1j} \geq 0$ and the point **L** belongs to $\mathbf{S}^+$. Hence the boundary condition (31) is approximated by

$$\frac{\mu}{a\Delta\lambda\sin\theta_j}(T_{oj} - T_{1j}) = 0 \qquad (50)$$

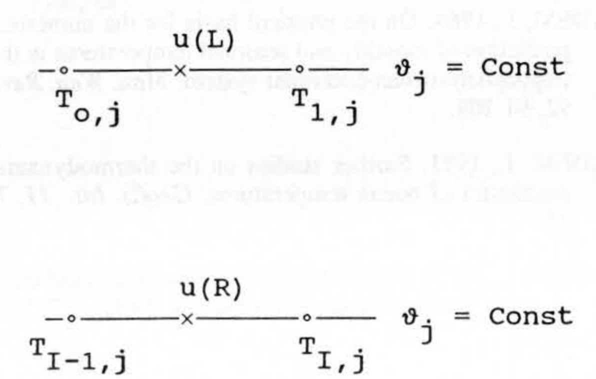

Fig. 4. Location of the grid nodes immediately adjacent to the boundary points **L** and **R** on a line θ_j = Const.

and hence, $T_{1j} = T_{oj}$. Thus the value T_{oj} of the function T in operator (46) for i=1 must be changed to T_{1j} so as to exclude the external point (λ_o, θ_j).

Further, if $u_{Ij} \equiv u(\mathbf{R}) \leq 0$ then $U_n = u_{Ij} \leq 0$, and the boundary point **R** belongs to $\mathbf{S}^-$. Furthermore

$$\frac{\partial T}{\partial n}(\mathbf{R}) \cong \frac{1}{a\Delta\lambda\sin\theta_j}(T_{Ij} - T_{I-1,j}) . \qquad (51)$$

and hence, the boundary condition (30) is approximated as

$$\frac{\mu}{a\Delta\lambda\sin\theta_j}(T_{Ij} - T_{I-1,j}) - \frac{1}{2}u_{I,j}(T_{I-1,j} + T_{Ij}) = 0 . \qquad (52)$$

Finally, if $u_{Ij} \equiv u(\mathbf{R}) \geq 0$ then the boundary point **R** belongs to $\mathbf{S}^+$, and (31) is reduced to

$$T_{Ij} = T_{I-1j} . \qquad (53)$$

For the boundary points located on the lines λ_1= Const, the boundary conditions are approximated in the same manner. Of course, the number of inner grid points varies from line to line. We see that the differential equation (1), and the boundary conditions (30)-(31) are approximated to second order in the geometric variables.

9. NUMERICAL SCHEME OF THE MODEL

Recalling (49), (50), (52) and (53), we can show that each of the matrices A_m^h is positive semidefinite:

$$T^T A_m^h T \geq 0 , \ (m = 1, 2) . \qquad (54)$$

Here T is the column vector with components T_{ij}, and the superscript T is the transposition symbol. Thus the finite-difference approximations (46) and (47) conserve the positive semidefiniteness of the differential operators A_m (m=1, 2). Therefore the application of the splitting method in the construction of numerical schemes is justified.

Suppose that we are required to solve the equation (1) within a given time interval. The interval is divided into equal subintervals (t_{n-1}, t_n) of small length $\tau = t_n - t_{n-1}$. Since the matrices A_i^h do not commute, the symmetric variant of the splitting method described at the end of section 7 will be applied to obtain the *2nd* order difference scheme within each double subinterval $I_n \equiv (t_{n-1}, t_{n+1})$:

$$T\left[n-\tfrac{1}{2}\right] - T[n-1] = -\tfrac{\tau}{2}A_1^h\left\{T\left[n-\tfrac{1}{2}\right] + T[n-1]\right\}$$

$$T\left[n+\tfrac{1}{2}\right] - T\left[n-\tfrac{1}{2}\right] = -\tau A_2^h\left\{T\left[n+\tfrac{1}{2}\right] + T\left[n-\tfrac{1}{2}\right]\right\} + 2\tau\ F[n]$$

$$T[n+1] - T\left[n+\tfrac{1}{2}\right] = -\tfrac{\tau}{2}A_1^h\left\{T[n+1] + T\left[n+\tfrac{1}{2}\right]\right\} . \qquad (55)$$

Here T[n-1] and T[n+1] are the column vectors with the components $T_{ij}(t_{n-1})$ and $T_{ij}(t_{n+1})$, respectively; $F[n] = \frac{1}{2}\{F_{ij}(t_{n+1})+F_{ij}(t_{n-1})\}$ is the forcing approximation; and T[n-1/2] and T[n+1/2] are the auxiliary vectors (n = 1, 3, 5,...). Each of the 1-D equations in (55) is solved by using the Crank-Nicholson scheme. The solution T[n-1] for the previous interval I_{n-1} is taken as the initial condition for the interval I_n. The initial vector T[0] has components $T^0(\lambda_i,\theta_j)$ (see Eq.(4)).

Multiplying the first equation (55) by the vector $\left\{T\left[n-\frac{1}{2}\right]+T[n-1]\right\}^T$ from the left, and taking into account (54) we find

$$T^T\left[n-\frac{1}{2}\right]T\left[n-\frac{1}{2}\right]\leq T^T[n-1]T[n-1] \ . \qquad (56)$$

If $F[n] \equiv 0$ then the inequalities such as (56) are valid for each of the three equations (55). Since $T^TT = \|T\|^2$ where $\|T\|$ is the Euclidean norm, these inequalities lead to

$$\|T[n+1]\| \leq \|T[n-1]\| \ , \qquad (57)$$

that is, scheme (55) is stable to initial perturbations for any time step τ.

Let us multiply equations (55) from the left by the row vector V^T whose components V_{ij} equal the values $a^2\Delta\lambda\,\Delta\theta\sin\theta_j$ in the inner nodes of the grid domain. Summing all results yields a finite-difference variant of the balance equation (35). Hence the scheme (55) is balanced. Moreover, when μ=0 for the forcing F[n]=0 and U_n≡0 at the boundary **S**, then the scheme (55) has the two conservation laws:

$$\sum_{i,j} T_{ij}(t_{n+1})V_{ij} = \sum_{i,j} T_{i,j}(t_{n-1})V_{ij} \qquad (58)$$

$$\|T[n+1]\| = \|T[n-1]\| \ . \qquad (59)$$

Although the scheme (55) is absolutely stable for an arbitrary time step τ, its magnitude is evidently limited by the requirements of the approximation. For the original equation (1), the approximation problem was discussed by Adem (1971). The splitting method imposes additional restrictions on the choice of τ. Numerical experiments by Marchuk and Skiba (1976, 1978, 1992) with the global thermodynamic model show that a 6-hour time step will provide a good approximation for the scheme (55). This scheme is solved without iterations by the direct method (factorization), and can be readily generalized to the 3-D heat transport model (Skiba, 1993).

10. CONCLUSIONS

Two mathematical problems have been formulated for the Adem ocean thermodynamic model intended for calculating the SST anomalies in closed or open oceanic basins. Special attention has been given to setting physically and mathematically appropriate boundary conditions (30)-(32) for the open basin model, where anomalous heat flow takes place across the boundaries. These conditions can be obtained from the generalised state of the problem. In the nodiffusive limit (μ=0), the boundary conditions are transformed to natural conditions for the pure advection problem. As a result, both problems are well-posed in the sense of Hadamard, that is, either solution is unique and stable to initial perturbations. Since the model operator is non-negative, application of the splitting method in the construction of finite-difference schemes is justified. This method reduces the original 2-D problem to solving a few simple 1-D problems. The implicit numerical algorithm does not require iterations: it is solved by the direct method of factorization and is readily generalized to 3-D heat transport problems.

The numerical scheme is implicit, balanced, unconditionally stable, of second-order approximation in the space and time variables, and affordable even for the 3-D problem. It has two conservation laws (see (58) and (59)) when F=0, and the model operator is skew-symmetric, i.e., in the nondiffusive case (μ = 0) when the orthogonal projection of the velocity vector on the outer normal is zero at the boundary of the basin. The spherical coordinate system permits the numerical algorithm to be applied to an oceanic basin Ω of arbitrary form, size and location.

A method based on the solution of the eigenvalue problem has been suggested to estimate the rate of decay of SST anomalies in the absence of heat forcing. The method is applicable for an oceanic domain Ω of arbitrary shape. It can also be used for selecting a model diffusion coefficient μ such as to bring the rate of dissipation of the SST anomalies given by the model into coincidence with the observed values in Ω.

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