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Fast potential field modelling based on digital filtering

Uwe Koppelt and Javier Rojas
Universität Leipzig
Institut für Geophysik und Geologie, Leipzig, Germany.

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RESUMEN

En los últimos años se han publicado una serie de artículos que se ocupan de la modelización de campos potenciales. Aquí se discuten las desventajas de algunos métodos que hacen uso de expresiones analíticas para el campo potencial.

Se presenta un algoritmo universal para la modelización de cuerpos discretos de regular construcción. Esto se basa en el teorema discreto de convolución. Se describe un procedimiento para eliminar los efectos de contorno en los cálculos. Para acelerar la computación de este procedimiento se utilizan las técnicas de la transformada de Fourier Rápida.

PALABRAS CLAVE: Modelización rápida, campos potenciales, transformada de Fourier.

ABSTRACT

In recent years many papers dealing with potential field modelling have been published. The main disadvantages of some methods using analytical expressions for potential field spectra are considered.

A universal modelling algorithm for potential fields of regularly constructed discrete models is presented. It is based on the discrete convolution thorem. To insulate the output from artifacts of undesired periodicity a complete treatment of edge effects is described. In order to speed up the computation, the fast Fourier transform and its symmetries are utilized. We are dealing with a discrete convolution of two well behaved functions; more over, one of them having final support. By computing Green's function accurately, no other errors except for round-off errors can affect the data within the desired output range.

KEY WORDS: Fast potential field modelling, FFT.

INTRODUCTION

Fundamentally, the calculation of gravitational and magnetic effects of subsurface structure rests on general attraction laws of potential fields. As is well known, the gravity as well as other potential fields of arbitrary structures can be expressed as a 3-D convolution of a Green's function (representing the effect of the point source) with the distribution function of physical parameters within the structure-the source strength. According to the convolution theorem, the Fourier transform of the convolution is just the product of the individual Fourier transforms (Bracewell, 1965).

So, it seems to be very easy to compute potential fields efficiently using the fast Fourier transform (FFT). But, things are not always what they seem to be. To solve the problem in a reasonable way, an understanding of digital filtering is very important. Mainly the relations between discrete and continuous Fourier transform and the convolution theorem should be kept in mind.

In the last three decades considerable attention has been devoted to finding an optimal algorithm from the viewpoint of accuracy and speed of computation. Many papers have been published, everyone of them promising a new and effective method for computing the potential field of several geological bodies.

The most used algorithm, especially as a tool in solving inverse problems, was developed by Parker (1973). He made use of a Taylor series expansion to express the potential field due to a layer with varying top and bottom topography. Bhattacharyya and Navolio (1975) derived a modelling algorithm for arbitrary bodies using descomposition into homogeneous prisms and the continuous convolution theorem to compute the field of individual prisms. A potential field expression for arbitrary polyhedrons in the frequency domain was derived by Pedersen (1978) using transformation of volume integrals into surface integrals, and later simplified by Hansen and Wang (1988).

Disregarding particular advantages of individual algorithms, there are mainly two disadvantages inherent to all the above-mentioned methods. First, the analytical expressions derived for the spectra are not optimal from numerical point of view. Usually the occurrence of infinite series, transcendental functions or the possible arisal of singularities is rather cumbersome. Second, they require computation of the inverse Fourier transform of a function having generally infinite support. Conversion into the spatial domain is done by the inverse FFT, and therefore replacing the continuous inverse Fourier transform by the inverse discrete Fourier transform (IDFT) leads to aliasing in the spatial domain.

Especially the second fact, well known from digital filtering, has never been discussed in the above-mentioned papers. An excellent approach to overcome this problem with minimum numerical expense was given by Chai and Hinze (1988).

Other procedures treating potential field modelling in a strictly numerical way (Meskó, 1977; Sideris, 1985; Bezvoda, 1987), that means, using the discrete convolution theorem, have proved to be more promising. Their one and only shortcoming is a low order approximation of the Green's function and, partly, an incomplete treatment of edge effects.

This paper deals with a universal modelling algorithm for potential fields of regularly constructed discrete models. The relations between potential field modelling and digital filtering are considered and some numerical aspects are discussed. To simplify matters all formulae are derived for the 2-D case. The extension to 3-D problems is quite simple.

FUNDAMENTAL RELATIONS

Any potential field anomaly $g(\vec{p})$ can be expressed as a 3-D convolution

$$g(\vec{p}) = \iiint F(\vec{p}-\vec{p}')\rho(\vec{p}')d^3\vec{p}' \tag{1}$$

of the Green's function F (representing the effect of a point source) with the distribution function of physical parameters within the source region - the source strength ρ. Herein $\vec{p}$ denotes the 3-D position vector. For 2-D structures integration over the co-ordinate in strike direction, e.g. y, can be performed analyticaly, and (1) changes to

$$g(\vec{r}) = \iint F(\vec{r}-\vec{r}')\rho(\vec{r}')d^2\vec{r}', \tag{2}$$

where $\vec{r} = \sqrt{x^2+z^2}$ denotes the position vector within a plane perpendicular to the strike of the structure under consideration.

In praxis, we are often interested in computing a finite number of potential field data at distinct points on a horizontal plane $\vec{r}_i$, $i = 1,\ldots, N$. The continuous problem can be discretized with the assumption that the source strength can be represented by a finite number M of coefficients, that is,

$$\rho(\vec{r}) = \sum_{j=1}^{M} \rho_j \varphi_j(\vec{r}) \ . \tag{3}$$

A widely used assumption is that the source is homogeneous within a number of subregions (elementary bodies), that means

$$\varphi_j(\vec{r}) = \begin{cases} 1 & : \ \vec{r} \in \Sigma_j \\ 0 & : \ \vec{r} \notin \Sigma_j \end{cases}, \quad j = 1,\ldots,M \tag{4}$$

Herein Σ_j denotes the area of cross-section of the j-th subregion. Inserting (3) into (2) leads to

$$g_i = \iint F(\vec{r}_i - \vec{r}') \sum_{j=1}^{M} \rho_j \varphi_j(\vec{r}') d^3\vec{r}' \tag{5}$$

$$= \sum_{j=1}^{M} F_{i-j}\rho_j \qquad i = 1,\ldots,N \tag{6}$$

where

$$F_{i-j} = \iint_{\Sigma_j} F(\vec{r}_i - \vec{r}')d^2\vec{r}' \tag{7}$$

Assuming Σ_j, $j = 1,\ldots, M$ to be a polygonal cross-section, formula (6) in connection with (7) can be easily solved using, e.g., the well known algorithm of Talwani (Talwani *et al.*, 1959). This method may be particularly appropriate when results of reflection seismic survey are at hand. But, it is not well suited for all problems. If the geological structures that have to be modelled are very complex, the computational effort increases rapidly. Such a situation may occur, if geophysical measurements for environmental studies or archeometry have to be interpretated.

That is why the convolution model has to be employed. As it can be seen, the discrete Green's function (7) represents the impulse response of a linear filter. Thus, equation (6) can be considered as a finite discrete convolution. In the special case of the source presumed to be composed of equal subregions (e.g., rectangular prisms) regularly distributed within several horizontal layers of arbitrary thickness as shown in Figure 1, (6) changes to

$$g_i = \sum_{k=1}^{K}\left(\sum_{j=1}^{N} F_{i-j,k}\rho_{j,k} \right), \quad i = 1,\ldots,N \tag{8}$$

Herein K and N denote the number of layers and subregions whithin the layer, respectively.

Assuming further the observation points to be nodes of a regular grid with constant elevation corresponding to discretization of the source region in the horizontal plane, $\{F_{i-j,k}\}$ becomes a number of shift-invariant filters. Application of the discrete convolution theorem (see, e.g., Elliot, 1987) leads to

$$G_n = \sum_{k=1}^{K} \Phi_{n,k} \cdot R_{n,k} \ , \quad n = 1,\ldots,N \tag{9}$$

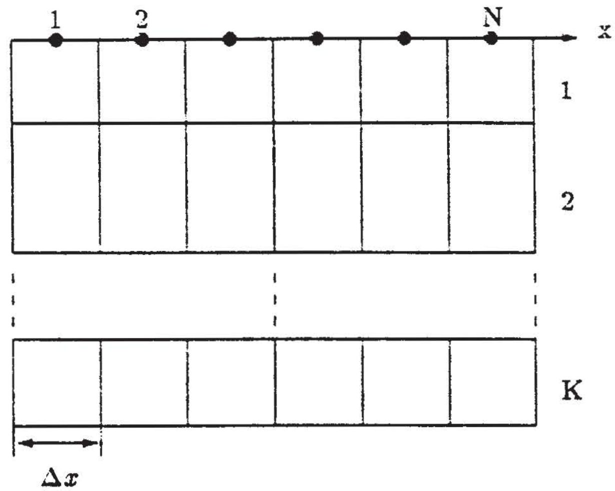

Fig. 1. 2-D model for computation of gridded potential field data (• denotes nodes of the grid).

where $\{G_n\}_{n=1}^{N}$ denotes discrete Fourier transform (DFT) of the set of potential field data $\{g_i\}_{i=1}^{N}$ $\{\Phi_{n,k}\}_{n=1}^{N}$ and $\{R_{n,k}\}_{n=1}^{N}$ denote DFTs of the Green's function and source strength for the k-th layer, respectively.

COMPUTATION OF THE GREEN'S FUNCTION

The Green's function is determined by the choosen elementary body, which depends on the nature of the individual potential field as well as on the desired accuracy. For example, in gravity or magnetic field modelling the elementary body is often represented by a rectangular prism. The corresponding expression for Green's function can be computed with the aid of formula (7). Thus, for the gravity and magnetic field (measured at horizontal plane $z = 0$), respectively, one obtains

$$\Delta g(x_n) = \gamma \sum_{l=1}^{2} \sum_{m=3}^{4} (-1)^{l+m} \left[(b_m - x_n)\ln r_{n,l,m} - b_l \arctan\left(\frac{b_l}{b_m - x_n} \right) \right] \quad (10)$$

$$\Delta T(x_n) = c_1 \sum_{l=1}^{2} \sum_{m=3}^{4} (-1)^{l+m} \left[c_2 \ln r_{n,l,m} + c_3 \arctan\left(\frac{x_n - b_m}{b_l} \right) \right] \quad (11)$$

where

$$x_n = n \cdot \Delta x, \qquad n = 1, \ldots, N, \quad (12)$$

$$\vec{b} = \left(z_1, z_2, -\frac{\Delta x}{2}, \frac{\Delta x}{2} \right), \quad (13)$$

$$r_{n,l,m} = \sqrt{(b_m - x_n)^2 + b_l^2}, \quad (14)$$

$$c_1 = \sin(2i'), \quad (15)$$

$$c_2 = \cos(2i'), \quad (16)$$

$$c_3 = \frac{T_0}{2\pi}\left(\cos^2 i \cos^2 \lambda - 1 \right), \quad (17)$$

$$i' = \arctan\left(\frac{\tan i}{\sin \lambda} \right). \quad (18)$$

Herein γ, λ, i and T_0 denote the gravity constant, the angle between north direction and strike of the 2-D structure, the inclination and the undisturbed Earth's magnetic field, respectively. The dimensions of the elementary body, as its top and bottom depth as well as its left and right border, are described by the parameter vector b.

If the Green's function varies only slowly with position within the elementary body, it holds approximately that

$$\hat{F}_{i-j} = \iint F(\vec{r}_i - \vec{r}\,')\varphi_j(\vec{r}\,')d^2\vec{r}\,' \approx F(\vec{r}_i - \vec{r}_j)\Sigma_j, \quad (19)$$

respectively,

$$\tilde{F}_{i-j} = \iint_{\Sigma_j} F(\vec{r}_i - \vec{r}\,')d^2\vec{r}\,' \quad (20)$$

$$= \int_{x_j+\Delta x/2}^{x_j+\Delta x/2} \int_{z_1}^{z_2} F(x_i - x', z')dz'dx' \quad (21)$$

$$= \int_{x_j+\Delta x/2}^{x_j+\Delta x/2} [FS(x_i - x', z_2) - FS(x_i - x', z_1)]\, dx' \quad (22)$$

$$\approx \left[FS(x_i - x_j, z_2) - FS(x_i - x_j, z_1) \right] \cdot \Delta x \cdot w_j, \quad (23)$$

where $\vec{r}_j$ and FS denote the center of mass of the elementary body and the integral of function F over z', respectively. The weights w_j depend on the choosen method of numerical integration (for discussion see Bezvoda *et al.*, 1992).

In the sense of numerical mathematics the first approach is equivalent to discretization of the integral (2), and, the second approach corresponds to performing analytic integration in the vertical direction and numerical integration using the trapezoid rule in the horizontal direction. From the viewpoint of geophysical modelling the first approach describes the representation of real structures with the aid of point sources (3-D case) or horizontal straight lines (2-D case), whereas, the second approach corresponds to representation of real structures using vertical line sources passing through the center of mass of the elementary body (3-D case) or infinite vertical stripes (2-D case).

The first concept in connection with digital filtering was originally followed by Meskó (1977), the second by Bezvoda (1987). Both methods have been proved to be very fast, since the approximate Green's function can be computed efficiently. But on the other side, they are restricted to sources at depth greater than the grid spacing because of the variations of continuous Green's function are presumed to be very small within every elementary body. In general, they cannot be applied to interpretation of geophysical measurements for environmental studies or in search for buried antiquities. Applied to anomalous source strength distribution located near the surface, these modelling techniques lead to great errors.

Figure 2 gives a comparison of accuracy of the three methods (herein a 3-D example is analyzed). The gravity of a prism with top and bottom depth of 0.5 m and 1.5 m, respectively, and density ρ = 1.0 g cm^{-3} was computed at nodes of a 2 m x 2 m grid. The prism was subdivided into 9 subprisms arranged parallel to each other. Employing the method of Meskó (1977), each subprism will be approximated by locating a point mass at its center. Bezvoda's (1987) method uses approximation of each subprism with the aid of a vertical line element through the center of mass. Figure 2a shows the gravity field of the whole prism computed using program NEW3D published by Nagy (1988). Figure 2b-d represent procentual error of computed gravity field using (b) the herein developed method, (c) Bezvoda's (1987) method and (d) Meskó's (1977) method. For the field is symmetric with respect to co-ordinate axis, only the first square is drawn. This clearly shows the advantage of the described method.

THE TREATMENT OF EDGE EFFECTS

From the viewpoint of digital filtering, formula (9) represents a number of discrete convolutions of a "signal" or data set with a "response function" of a digital filter. Herein, the Green's function and the source strength denote respectively the signal and filter.

The discrete convolution theorem presumes two circumstances which are not universal. First, it assumes that the input signal is periodic. Second, the convolution theorem takes the duration of the response to be the same as the period of the signal.

Since we are chiefly interested in a response function whose duration M is shorter than the length of the data set N, this problem can be solved very easily. One simply extends the response function to length N by padding it with zeros.

The first one is a little bit more complicated. After Kanasewich (1981), convolution can be considered as a "sliding" of the filter inverted in space over the signal. Since the convolution theorem assumes the data to be periodic, it will falsely compute the first $M_{(+)}$ output values with some wrapped-around data from the far end of the data set, where $M_{(+)}$ denotes the number of so-called positive indices of the filter. The same holds for the last $M_{(-)}$ output values. Correspondingly, $M_{(-)}$ denotes the number of so-called negative indices of the filter. So, we have to extend the discrete Green's function by $M_{(+)}$ values at the begining and by $M_{(-)}$ at the end to set a buffer zone.

Assuming the filter coefficient $\rho[N/2],k$ to correspond with spatial co-ordinate x=0, it holds that

$$M_{(+)} = \left[\frac{N}{2}\right], \tag{24}$$

$$M_{(-)} = \left[\frac{N}{2}\right] - 1, \tag{25}$$

where $[N/2]$ denotes the greatest integer value of $N/2$.

Note that the response function, zero padded out to the new duration N_1 - N+ $(M-1)$, should be arranged in wrap-around order. Generally this means the following arrangement:

$$\rho[N_1/2],k,\ \rho[N_1/2]+1,k,\ldots,\ \rho[N_1,k,\rho 1,k,\ldots,\ \rho[N_1/2]-1,k,\quad k=1,\ldots,K. \tag{26}$$

Combining these operations (see Figure 3), we effectively insulate the output from artifacts of undesired periodicity in the desired output range.

USEFULNESS OF THE FAST FOURIER TRANSFORM

The modelling of potential fields in the frequency domain rather than in the spatial domain is appropriate for the processing of large data sets. Because the DFT can be computed in $O(N \log_2 N)$ operations with the aid of the FFT, the computation of potential fields using equation (9) requires $O(3K{\cdot}N\log_2 N)$ operations instead of $O(K{\cdot}N^2)$ operations required by (8). Therefore, to speed up the numerical evaluation of spectral representation of both the source strength and the Green's function, the symmetry of the DFT can be used to compute the transform of the two real functions ρ and F simultaneously.

Let us introduce the auxiliary complex variables $z[n]$ by the notation

$$z_n = \rho_n + iF_n, \quad n = 1,\ldots,N_1. \tag{27}$$

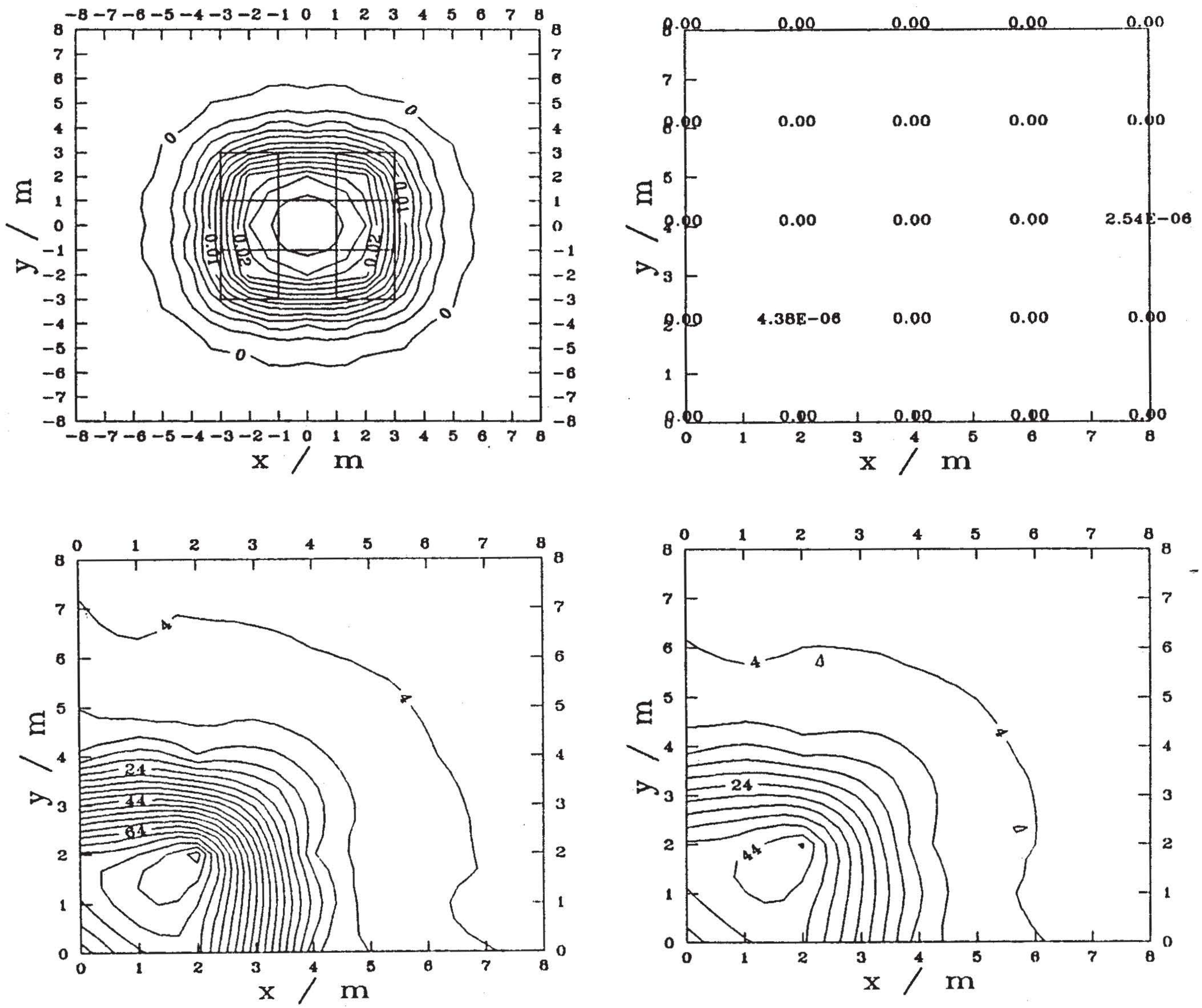

Fig. 2. Comparison of accuracy of modelling algorithms utilizing the discrete convolution theorem. (a) synthetic model and its gravitational attraction computed using program NEW3D (Nagy, 1988); percentual error distribution of (b) the described algorithm, (c) Bezvoda's (1987) method and (d) Meskó's (1977) method, respectively.

From the reality of the functional values ρ_n and F_n it follows, when the convention

$$R_{N_1+1} = R_1, \quad \Phi_{N+1} = \Phi_1 \tag{28}$$

is used

$$R^*_m = R_{N_1-m+2}, \quad \Phi^*_m = \Phi_{N_1-m+2} \tag{29}$$

and, consequently,

$$\Re R_m = \frac{1}{2}(\Re Z_m + \Re Z_{N_1-m+2}) \tag{30}$$

$$\Im R_m = \frac{1}{2}(\Im Z_m - \Im Z_{N_1-m+2}) \tag{31}$$

$$\Re \Phi_m = \frac{1}{2}(\Im Z_m + \Im Z_{N_1-m+2}) \tag{32}$$

$$\Im \Phi_m = -\frac{1}{2}(\Re Z_m - \Re Z_{N_1-m+2}) \tag{33}$$

$$m = 1,\ldots,N_1 \tag{34}$$

where Z_m, R_m and Φ_m are the DFTs of the sets z_n, ρ_n and F_n, respectively. $\Re$ and $\Im$ denote respectively real and imaginary part of a complex number.

As convolution in space domain is equivalent to multiplication in frequency domain, thus

$$G_m = R_m \cdot \Phi_m, \quad m = 1,\ldots,N_1 \tag{35}$$

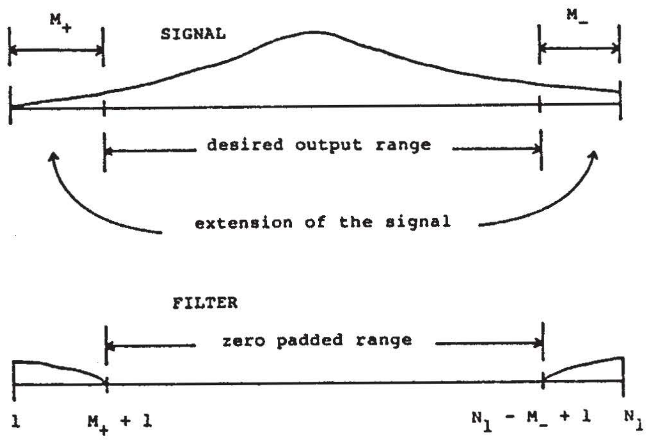

Fig. 3. Storage arrangement of the Green's function (signal) and the source strength (filter).

where G_m denotes the DFT of the potential field. It follows that

$$\Re G_m = \frac{1}{2}(\Re Z_m \cdot \Im Z_m + \Im Z_{N_1-m+2} \cdot \Re Z_{N_1-m+2}) \quad (36)$$

$$\Im G_m = -\frac{1}{4}\left((\Re Z_m)^2 - (\Im Z_m)^2 - (\Re Z_{N_1-m+2})^2 + (\Im Z_{N_1-m+2})^2 \right) . \quad (37)$$

Then the potential field can be easily obtained by application of the inverse FFT to the set G_m.

CONCLUSIONS

In this paper a universal algorithm for potential field modelling based on digital filtering is presented. Using the discrete convolution theorem, the problem is transformed into the frequency domain where convolution changes to simple multiplication. To speed the calculation process, the FFT and its symmetries are utilized. A technique to treat edge effects is completely described. No other errors except for round-off errors and those propagated from the approximation of the Green's function can affect the data within the desired output range.

The price we have to pay for simplicity and rapidity is that inherent to all Fourier transform techniques: potential field data are obtained only at nodal points of a grid corresponding to the discretization of the model on a single surface with constant elevation.

The algorithm is well suited for all interpretation techniques based on forward modelling, such as *trial and error, Monte Carlo method, Simulated annealing* (see, e.g., Tarantola, 1987) or algorithms based on *evolution strategy*. Notwithstanding its higher accuracy in comparison to conventional modelling techniques utilizing the convolutional model, the forward modelling algorithm works very fast. Thus, it can be done many times in a short time interval. The change of parameters as well as the choice of best fitting model depends on the employed interpretation technique.

BIBLIOGRAPHY

BEZVODA, V., 1987. Fast algorithms in geophysics and geochemics (in czech). Doctor thesis, Charles University, Prague.

BEZVODA, V., J. HRABE and K. SEGETH, 1992. Fast algorithm for solving potential field problems. *In:* Vogel A., Sarwar A. K. M., Gorenflo R., Kouchev O. J. (Eds.): Theory and practice of geophysical data inversion. Vieweg, Braunschweig.

BHATTACHARYYA, B. K. and M. E. NAVOLIO, 1975. Digital convolution for computing gravity and magnetic anomalies due to arbitrary bodies. *Geophysics 40,* 981-992.

BRACEWELL, R., 1965. The Fourier transform method and its application. McGraw-Hill, New York.

CHAI, Y. and W. J. HINZE, 1988. Gravity inversion of an interface above which the density contrast varies exponentially with depth. *Geophysics 53,* 837-846.

ELLIOT, D. F., 1987. Handbook of digital signal processing, Engineering applications. Academic press, London.

HANSEN, R. O. and X. WANG, 1988. Simplified frequency domain expression for potential fields of arbitrary three-dimensional bodies. *Geophysics 53,* 365-374.

KANASEWICH, E. R., 1981. Time sequence analysis in geophysics. University of Alberta Press, Edmonton.

MESKÓ, A., 1977. A new algorithm for the computing of gravitational attraction due to irregulary shaped bodies. *Ann. Univ. Budapest 18,* 65-73.

NAGY, D., 1988. A short program for three-dimensional gravity modelling. Acta Geod. *Geoph. Mont. Hung. 23,* 449-459.

PARKER, R. L., 1973. The rapid compution of potential anomalies. *Geophys. Journ. Roy. Astr. Soc. 31,* 447-455.

PEDERSEN, L. B., 1978. Wavenumber expression for potential fields from arbitrary 2-,2- and 3- dimensional bodies. *Geophysics 43,* 626-630.

SIDERIS, M. G., 1985. A fast Fourier transform method for computing terrain corrections. *Manuscripta Geodaetica 10*, 66-73.

TALWANI, M., J. L. WORZEL and M. LANDISMAN, 1959. Rapid gravity computation for two-dimensional bodies with application to the Mendocino submarine fracture zone. *J. Geophys. Res. 64*, 49-59.

Uwe Koppelt and Javier Rojas
Universität Leipzig
Institut für Geophysik und Geologie, Talstrasse 35, 0-7010 Leipzig, Germany.