Geofísica Internacional (1994), Vol. 33, Num. 4, pp. 541-545

Axi-symmetric eigenmodes for an elastic semi-infinite circular cylindrical rod

Pratap N. Sahay[1] and Anton Z. Capri[2]
[1] *Centro de Investigación Científica y Educación Superior de Ensenada, Ensenada, México.*
[2] *Department of Physics, University of Alberta, Edmonton, Canada.*

Received: September 24, 1993; accepted: March 11, 1994.

RESUMEN
Construimos un conjunto completo de ejes simétricos para el campo de desplazamientos dentro de una barra elástica, homogénea, isotrópica y semi-infinita con condiciones de frontera semi-rígidas sobre la superficie curva de la barra y condiciones de frontera libre de esfuerzo sobre la superficie plana de la misma.

PALABRAS CLAVE: Cilindro circular, autofunciones.

ABSTRACT
We have constructed a complete orthonormal set of axi-symmetric eigenmodes for the displacement field inside an elastic, homogeneous and isotropic semi-infinite cylindrical rod with semi-rigid boundary conditions on the curved surface of the rod and stress-free boundary conditions on the flat surface capping the rod.

KEY WORDS: Circular cylinder, eigenmodes.

INTRODUCTION

The problem under consideration belongs to a class of problems with two level surfaces. This geometry complicates the computation due to mode conversion at the boundaries. For rigid or stress-free surfaces both longitudinal and shear waves undergo mode conversion at these surfaces and such problems are in general nonseparable. However, if at one of the surfaces semi-rigid boundary conditions are imposed, the problem is still separable. Semi-rigid boundary conditions mean that:

1) The normal component of displacement vanishes and
2) The tangential component of stress vanishes.

For this reason we have chosen these boundary conditions on the curved surface of the rod. The end of the rod (flat surface) satisfies stress-free boundary conditions. The eigenmodes thus obtained form a complete set and may be used to solve problems even with stress-free or rigid boundary conditions.

We have introduced the semi-rigid boundary condition because it is one of three possible boundary conditions that render self-adjoint the differential operator $D(\partial)$ in

$$D(\partial)\mathbf{u} = C_p^2 \nabla(\nabla \cdot \mathbf{u}) - C_s^2 \nabla \mathrm{x} (\nabla \mathrm{x} \mathbf{u}) \tag{1.1}$$

where C_p and C_s are the constant P and S wave velocity (Sahay and Capri, 1988). The other two boundary conditions yielding self-adjointness are stress-free and rigid. The semi-rigid boundary condition is not only of academic interest. It also corresponds to interesting physical situations, for example, when one has a soft medium abutting against a stiff material. At a surface with semirigid boundary conditions only reflection without mode-conversion occurs. Thus, for a problem with many level surfaces, it is possible to use this boundary condition to suppress mode conversion.

We have also solved the corresponding exterior problem for a cylinder imbedded in a elastic halfspace (Sahay and Capri, 1989). Using that result plus our present work makes it possible to solve the elastodynamic problem for a cylinder imbedded in an elastic half-space with different elastic parameters. For these problems the sources that can be handled must feature the same cylindrical symmetry as the imbedded cylinder. Thus, we may have cylindrical sources, ring sources, or even a point or line source on the axis of the cylinder. This may provide a sensible starting point for constructing models for geological problems such as magmatic intrusion. It should also be of interest to various engineering applications such as concrete piles imbedded in the ground. Kim and Steele (1989) have studied longitudinal waves in a semi-infinite cylinder with stress-free boundary conditions by expanding the solution in terms of eigenfunctions for a semi-rigid infinite cylinder. The complete set of eigenfunctions constructed here may be appropriate for their problem because the stress-free boundary condition at the end surface capping the semi-infinite cylinder has already been incorporated.

Throughout this discussion we are assuming that the medium is homogeneous and isotropic.

In section 2 we present the mathematical formulation. The results are presented in section 3. Since the normalization and completeness for this problem are not entirely straightforward, we sketch them in section 4. The final section contains the conclusions.

FORMULATION OF THE PROBLEM AND DECOMPOSITION OF THE WAVE EQUATION

Consider a displacement field $\mathbf{u}(\mathbf{r}, t)$ in the interior of a semi-infinite homogeneous, isotropic circular rod whose surfaces are specified by

$$0 \leq r \leq a,\ 0 \leq \theta < 2\pi,\ 0 \leq z < \infty.$$

The displacement field satisfies the elastic wave equation

$$C_p^2 \nabla(\nabla \cdot \mathbf{u}) - C_s^2 \nabla \mathrm{x}(\nabla \mathrm{x} \mathbf{u}) = \frac{\partial^2 \mathbf{u}}{\partial t^2}\ . \qquad (2.1)$$

The boundary conditions satisfied at the cylindrical surface of the rod are as follows:

1) At the curved surface of the rod r=a we impose semi-rigid boundary conditions. These read

$$u_r \big|_{r=a} = 0 \qquad (2.2)$$

$$\sigma_{r\theta} \big|_{r=a} = 0 \qquad (2.3)$$

$$\sigma_{rz} \big|_{r=a} = 0 \qquad (2.3)$$

2) At the flat surface of the rod z=0 we impose stress-free boundary conditions. These read

$$\sigma_{zr} \big|_{z=o} = 0 \qquad (2.5)$$

$$\sigma_{z\theta} \big|_{z=o} = 0 \qquad (2.6)$$

$$\sigma_{zz} \big|_{z=o} = 0 \qquad (2.7)$$

We are interested in finding time-harmonic solutions since a general time dependence can be constructed from these by superposition. Thus, we set $\mathbf{u}(\mathbf{r},t) = \mathbf{u}(\mathbf{r})e^{-i\omega t}$ and replace this in equation (2.1). This wave equation now reads

$$D(\partial)\,\mathbf{u} = -\,\omega^2\,\mathbf{u} \qquad (2.8)$$

where the operator $D(\partial)$ is defined in equation (1.1). As stated in the introduction, the boundary conditions (2.2) - (2.7) plus the radiation condition render this differential operator self-adjoint. Thus, we are guaranteed the existence of a complete set of eigenmodes.

To solve equation (2.8) we decompose the displacement field $\mathbf{u}$ by introducting the Lamé potentials (ϕ, ψ, χ) via

$$\mathbf{u} = \nabla\phi + \nabla \mathrm{x} \hat{z} \psi + l\, \nabla \mathrm{x} \nabla \mathrm{x} \hat{z} \chi \qquad (2.9)$$

where $\hat{z}$ is a unit vector in the z- direction and l is a convenient length scale factor. Equation (2.8) now decouples into three scalar Helmholtz equations

$$(\nabla^2 + K_p^2)\phi = 0 \qquad (2.10)$$

$$(\nabla^2 + K_s^2)\psi = 0 \qquad (2.11)$$

$$(\nabla^2 + K_s^2)\chi = 0 \qquad (2.12)$$

where $K_p = \omega/C_p$, $K_s = \omega/C_s$ and

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{\partial^2\phi}{\partial z^2}\ . \qquad (2.13)$$

Since the boundary conditions show cylindrical symmetry, the problem is essentially two-dimensional and therefore we have dropped the derivatives with respect to θ. In terms of potentials, the boundary conditions read:

$$u_r\big|_{r=a} = (\partial_r\phi + l\partial_r\partial_z\chi)\big|_{r=a} = 0 \qquad (2.14)$$

$$\sigma_{r\theta}\big|_{r=a} = -\mu(\partial_r^2\psi - 1/r\,\partial_r\psi)\big|_{r=a} = 0 \qquad (2.15)$$

$$\sigma_{rz}\big|_{r=a} = \mu[2\partial_r\partial_z\phi + l\partial_r(2\partial_z^2\chi - \nabla^2\chi)]\big|_{r=a} = 0 \qquad (2.16)$$

$$\sigma_{rz}\big|_{r=a} = \mu[2\partial_r\partial_z\phi + l\partial_r(2\partial_z^2\chi - \nabla^2\chi)]\big|_{z=o} = 0 \qquad (2.17)$$

$$\sigma_{z\theta}\big|_{z=o} = -\mu(\partial_r\partial_z\psi)]\big|_{z=o} = 0 \qquad (2.18)$$

$$\sigma_{zz}\big|_{z=o} = \left\{\lambda\nabla^2\phi + 2\mu\left[\partial_z^2\phi + l\partial_z(\partial_z^2\chi - \nabla^2\chi)\right]\right\}\big|_{z=o} = 0\ . \qquad (2.19)$$

The scalar potential ψ decouples completely from the other two potentials and its solution may be displayed inmediately.

$$\psi = AJ_o(kr)\cos\beta z \quad \text{where} \quad k \leq K_s\ , \qquad (2.20)$$

and β is defined further down by equation (2.25). We still have to satisfy the boundary condition at r=a. This condition forces the separation constant k to be discrete.

The potentials ϕ and χ remain coupled through the boundary conditions. Each of these potentials may be expressed as a product of a Bessel function or else an exponential function in the variable z. Since there are no singularities at the origin, only the ordinary Bessel function J_o is permitted. Thus, the general solutions are of the form:

$$\phi = J_o(kr)(A\sin\alpha z + B\cos\alpha z) \quad \text{for} \quad k \leq K_p\ , \qquad (2.21)$$

$$\phi = EJ_o(kr)e^{-\Gamma z} \quad \text{for} \quad k > K_p \qquad (2.22)$$

and

$$\chi = J_o(kr)(C\sin\beta z + D\cos\beta z) \quad \text{for} \quad k \leq K_z \qquad (2.23)$$

$$\chi = FJ_o(kr)e^{-\Delta z} \quad \text{for} \quad k > K_s \qquad (2.24)$$

where

$$\alpha = (K_p^2 - k^2)^{1/2}\ ,\ \beta = (K_s^2 - k^2)^{1/2} \qquad (2.25)$$

and

$$\Gamma = (k^2 - K_p^2)^{1/2}, \quad \Delta = (k^2 - K_s^2)^{1/2} \ . \qquad (2.26)$$

Here k is simply a separation variable whose possible values are again determinded by the boundary conditions at r=a. Imposing the boundary conditions at r=a yields:

$$J_1(ka) = 0 \qquad (2.27)$$

The allowed values of k are

$$k = k_n = x_n / a \qquad (2.28)$$

where x_n are the zeroes of $J_1(x)$. The values of α, β, Γ, and Δ for discrete values of k (that is, for k_n) are henceforth written as α_n, β_n, Γ_n and Δ_n. We now set l equal to $1/k_n$.

The classification given above leads to four classes of solutions. These are displayed and discussed in the next section.

RESULTS

The properly normalized eigenmodes are listed below. We list the potentials as well as the displacement fields. In the following θ denotes a Heaviside function.

Mode 1

$$\begin{bmatrix} \phi^{(1)} \\ \chi^{(1)} \end{bmatrix} = A^{(1)} J_o(k_n r) \begin{bmatrix} (\beta_n^2 - K_n^2)\sin\alpha_n z \\ 2k_n\alpha_n\cos\beta_n z \end{bmatrix} \theta(K_p - k_n) \qquad (3.1)$$

$$\begin{bmatrix} u_r^{(1)} \\ u_z^{(1)} \end{bmatrix} =$$

$$A^{(1)} \begin{bmatrix} k_n J_1(k_n r)\left\{-(\beta_n^2 - k_n^2)\sin\alpha_n z + 2\alpha_n\beta_n\sin\beta_n z\right\} \\ \alpha_n J_o(k_n r)\left\{(\beta_n^2 - k_n^2)\cos\alpha_n z + 2k_n^2\cos\beta_n z\right\} \end{bmatrix} \theta(K_p - k_n) \qquad (3.2)$$

$$\left[A^{(1)}\right]^{-2} = \frac{\pi}{4}\omega\alpha_n a^2\left[(\beta_n^2 - k_n^2)^2 + 4k_n^2\alpha_n\beta_n\right] J_o^2(k_n a) \qquad (3.3)$$

Mode 2

$$\begin{bmatrix} \phi^{(2)} \\ \chi^{(2)} \end{bmatrix} = A^{(2)} J_o(k_n r) \begin{bmatrix} 2k_n\beta_n\cos\alpha_n z \\ (\beta_n^2 - k_n^2)\sin\beta_n z \end{bmatrix} \theta(K_p - k_n) \qquad (3.4)$$

$$\begin{bmatrix} u_r^{(2)} \\ u_z^{(2)} \end{bmatrix} =$$

$$A^{(2)} \begin{bmatrix} -\beta_n J_1(k_n r)\left\{2k_n^2\cos\alpha_n z + (\beta_n^2 - k_n^2)\cos\beta_n z\right\} \\ k_n J_o(k_n r)\left\{-2\alpha_n\beta_n\sin\alpha_n z + (\beta_n^2 - k_n^2)\sin\beta_n z\right\} \end{bmatrix} \theta(K_p - k_n) \qquad (3.5)$$

$$\left[A^{(2)}\right]^{-2} = \frac{\pi}{4}\omega\beta_n a^2\left[(\beta_n^2 - k_n^2)^2 + 4k_n^2\alpha_n\beta_n\right] J_0^2(k_n a) \qquad (3.6)$$

Mode 3

$$\begin{bmatrix} \phi^{(3)} \\ \chi^{(3)} \end{bmatrix} = A^{(3)} J_o(k_n r)\theta(K_s - k_n)\theta(k_n - K_p) \mathrm{x}$$

$$\mathrm{x}\begin{bmatrix} 2k_n\beta_n(\beta_n^2 - k_n^2)\exp -\Gamma_n z \\ (\beta_n^2 - k_n^2)\sin\beta_n z - 4k_n^2\Gamma_n\beta_n\cos\beta_n z \end{bmatrix} \qquad (3.7)$$

$$\begin{bmatrix} \mathrm{u}_r^{(3)} \\ \mathrm{u}_z^{(3)} \end{bmatrix} = A^{(3)}\theta(K_s - k_n)\theta(k_n - K_p)\mathrm{x}$$

$$\mathrm{x}\begin{bmatrix} -\beta_n J_1(k_n r)\left\{2k_n^2(\beta_n^2 - k_n^2)\exp -\Gamma_n z + (\beta_n^2 - k_n^2)^2\cos\beta_n z + 4k_n^2\Gamma_n\beta_n\sin\beta_n z)\right\} \\ k_n J_o(k_n r)\left\{-2\Gamma_n\beta_n(\beta_n^2 - k_n^2)\exp -\Gamma_n z + (\beta_n^2 - k_n^2)\sin\beta_n z - 4k_n^2\Gamma_n\beta_n\cos\beta_n z)\right\} \end{bmatrix} \qquad (3.8)$$

$$\left[A^{(3)}\right]^{-2} = \frac{\pi}{4}\omega\beta_n a^2\left[(\beta_n^2 - k_n^2)^4 + 16k_n^4\Gamma_n^2\beta_n^2\right]J_o^2(k_n a) \tag{3.9}$$

Mode 4

$$\begin{bmatrix}\phi^{(4)}\\ \chi^{(4)}\end{bmatrix} = A^{(4)}J_o(k_n r)\begin{bmatrix}(\Delta_n^2 + k_n^2)\exp-(\Gamma_n z)\\ 2k_n\Gamma_n\exp-(\Delta_n z)\end{bmatrix}\delta(k_n - K_r) \tag{3.10}$$

$$\begin{bmatrix}u_r^{(4)}\\ u_r^{(4)}\end{bmatrix} = A^{(4)}\delta(k_n - K_r)\mathrm{x}$$

$$\mathrm{x}\begin{bmatrix}k_n J_1(k_n r)\left\{-(\Delta_n^2 + k_n^2)\exp(-\Gamma_n z) + 2\Gamma_n\Delta_n\exp(-\Delta_n z)\right\}\\ \Gamma_n J_o(k_n r)\left\{-(\Delta_n^2 + k_n^2)\exp(-\Gamma_n z) + 2k_n^2\exp(-\Delta_n z)\right\}\end{bmatrix} \tag{3.11}$$

$$\left[A^{(4)}\right]^{-2} = \omega\Delta_n a^2\left[(\Gamma_n^2 + k_n^2) + (\Delta_n^2 + k_n^2)\frac{\Gamma_n^2 - 2\Gamma_n\Delta_n}{\Delta_n^2}\right]J_o^2(k_n a) \tag{3.12}$$

We have explicitly written an expression for the Rayleigh mode (Mode 4), but this mode will in general be absent since the wavenumbers k_n are discrete and thus cannot simultaneously be a Rayleigh wavenumber (K_r) which is a root of the characteristic equation

$$(\Delta_n^2 + k_n^2)^2 = 4k_n^2\Gamma_n\Delta_n. \tag{3.13}$$

Thus, the Rayleigh mode will exist only for very special values of the material constants.

The self-adjointness of the differential operator guarantees that these modes form a complete set. Nevertheless we indicate an explicit proof of this assertion in the next section where we also discuss the normalization.

NORMALIZATION AND COMPLETENESS RELATION

The eigenmodes are normalized in the following manner:

$$<\mathbf{u}^{(m)}(k_n,\omega),\mathbf{u}^{(m')}(k_{n'},\omega')> = \delta_{m,m'}\delta_{n,n'}\delta(\omega-\omega') \tag{4.1}$$

where the inner product is defined by:

$$<\mathbf{u}^{(m)}(k_n,\omega),\mathbf{u}^{(m')}(k_{n'},\omega')> = \int_o^\infty dz\int_o^a rdr\Big[u_r^{(m)}(r,z;k_n,\omega)$$

$$u_r^{(m')}(r,z;k_{n'},\omega') + u_z^{(m)}(r,z;k_n,\omega)u_z^{(m')}(r,z;k_{n'},\omega')\Big] \tag{4.2}$$

Here m and m' label the four possible modes. If the Rayleigh mode is present these labels run from 1 to 4. If the Rayleigh mode is absent they run from 1 to 3. As stated earlier, the wavenumbers k_n are discrete and are given by $k_n=x_n/a$ where x_n is the nth zero of the Bessel $k_n=x_n/a$ function J_1. The z integration involved in the normalization is exactly the same as in the exterior problem previously discussed (Sahay and Capri 1989) and the result is the same yielding a delta function in (ω-ω'). The radial integration involves terms of the form

$$\int_o^a rdrJ_1(k_n r)J_1(k_{n'}r) \quad \text{or} \quad \int_o^a rdrJ_o(k_n r)J_o(k_{n'}r)$$

both of which are equal to $\frac{a^2}{2}J_o^2(k_n a)\delta_{n,n'}$. The orthogonality in the mode labels m and m' follows automatically. Thus modes 3 and 4 are orthogonal to each other as well as to modes 1 and 2 due to their non-overlapping ranges in the wavenumbers. Modes 1 and 2 also happen to be mutually orthogonal.

The completeness relation required is explicitly written as:

$$\sum_m\sum_n\int d\omega u_i^{(m)}(\mathbf{r};k_n,\omega)u_j^{(m)}(\mathbf{r}';k_n,\omega) = \delta_{i,j}\delta(\mathbf{r}-\mathbf{r}') \tag{4.3}$$

The order of integration over ω and summation over n may be interchanged. This changes the limits on the ω integration and leaves the limit on the summation over n unrestricted. As a consequence the ω integration may again be handled in exactly the same manner as in the exterior problem (Sahay and Capri, 1989). This produces the required delta function in (z-z').

The unrestricted summation over n leads to two different series, namely:

$$\frac{2}{a^2}\sum_n\frac{J_o(k_n r)J_o(k_n r')}{J_o^2(k_n a)} \quad \text{or} \quad \frac{2}{a^2}\sum_n\frac{J_1(k_n r)J_1(k_n r')}{J_o^2(k_n a)}$$

In both cases the summation is over the zeroes of J_1. Thus, the first of these sums is a Dini series and the second is a Fourier-Bessel series (Watson, 1952). Both of them produce the desired delta function, namely $1/r\delta(r-r')$.

CONCLUSION

The eigenmodes obtained here together with those obtained for the exterior problem (Sahay and Capri, 1989) are sufficient to compute the Green's function for any problem with a cylindrically symmetric source. This allows one to solve wave propagation problems in elastic media for such situations as a cylindrical rod vertically imbedded in a half space of a different material. However, the source must be axisymmetric. For the eigenfunctions obtained, the boundary condition imposed on the curved surface of the rod is semi-rigid; yet the contact between the rod and the rest of the half space may be welded because these eigenfunctions form a complete set for the geometry described.

ACKNOWLEDGEMENTS

Pratap N. Sahay would like to thank the Secretaría de Educación Pública of Mexico (SEP) for financial support and Anton Z. Capri would like to thank the Natural Science and Research Council of Canada (NSERC) for a research grant.

BIBLIOGRAPHY

KIM, Y. Y. and C. R. STEELE, 1989. End effects and time harmonic longitudinal wave propagation in a semi-infinite solid cylinder. *ASME Journal of Applied Mechanics, 56*, 334-346.

SAHAY, P. N. and A. Z. CAPRI, 1988. Elastodynamics of a quarterspace with one boundary semi-rigid and the other stress-free. *Geophysical Journal, 92*, 235-252.

SAHAY, P. N. and A. Z. CAPRI, 1989. Eigenmodes for an elastic halfspace surrounding a semi-rigid cylinder. *Wave Motion, 11*, 407-417.

WATSON, G. N., 1952. Theory of Bessel Functions, 2nd. Ed., Cambridge University Press.

Pratap N. Sahay[1] and Anton Z. Capri[2]

[1] *Departamento de Sismología, Centro de Investigación Científica y Educación Superior de Ensenada, Km. 107 Carretera Tijuana-Ensenada, A.P. 2732, 22830 Ensenada, Baja California, México.*

[2] *Department of Physics, University of Alberta, Edmonton, Alberta, Canada, T6G 2J1*