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Some special solutions of Rayleigh's equation and the reflections of body waves at a free surface

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RESUMEN

A partir de una nueva representación de las raíces de la ecuación de Rayleigh para todos los valores de la relación de Poisson ν, se deriva una nueva expresión analítica para la raíz doble. Esta relaciona de una manera simple a los ángulos especiales de Brewster, que aparecen para ondas longitudinales o transversales incidentes en una superficie libre. Al mismo tiempo las peculiaridades de los coeficientes de reflexión R_{PS} y R_{SP} son investigadas.

PALABRAS CLAVE: ecuación de Rayleigh, ondas superficiales, reflexión de ondas internas, relaciones críticas de Poisson.

ABSTRACT

A new representation of all roots of Rayleigh's equation for all values of Poisson's ratio ν is proposed. A new analytical expression for the double root is derived. It is found to be simply related to special Brewster angles, which occur for incident longitudinal or transversal waves at a stress-free surface. Some peculiarities of the reflection coefficients R_{PS} and R_{SP} are discussed.

KEY WORDS: Rayleigh's equation, surface waves, body wave reflection, critical Poisson ratios.

INTRODUCTION

Rayleigh's equation was discovered in the late 19th century (Rayleigh, 1885). Yet it still attracts attention in recent publications (e. g. Rahman and Barber, 1995; Nkemzi, 1997; Malischewsky, 2000). While the simple numerical availability of all kinds of roots of Rayleigh's equation is widely recognised, it may be interesting to look at the problem from a deeper point of view, in order to obtain some insight into the behaviour of elastic materials. The existence of critical dimensionless parameters will be shown to have a deeper and subtle cause, which has not been adequately understood until now. As a seismologist I am thoroughly convinced that we will not fully understand the Earth as a whole unless we can gain a more complete understanding of the complicated phenomena of elastic wave propagation on a small scale.

The formalism in Rahman and Barber (1995) has the disadvantage that there is no compact representation for the whole range of Poisson's ratio ν. Nkemzi (1997) attempted to overcome this drawback by using methods from the theory of complex functions. Unfortunately, Nkemzi's complicated final result is incorrect. In pointing out this situation, Malischewsky (2000) presented perhaps the simplest formulas for obtaining the roots of the cubic equation which can be derived from Rayleigh's equation by rationalization. These formulas are valid over the entire range of possible Poisson's ratios ($-1 \le \nu \le 0.5$). As in Rahman and Barber (1995), negative values of ν were included as they can actually occur in some materials (see *e.g.* Lakes, 1987). With these convenient results at hand we may discuss once more the different solutions of the Rayleigh equation in order to derive some new analytical relations concerning the remarkable point where complex solutions cease to exist. In this way, we shall be able to demonstrate once again the close connections between surface and body waves.

SOME SPECIAL SOLUTIONS OF RAYLEIGH'S EQUATION

The well-known cubic equation that follows from Rayleigh's equation by rationalisation is

$$x^3 - 8x^2 + 8x(3-2\gamma) - 16(1-\gamma) = 0 \quad \text{with } x = c^2/\beta^2 \tag{1}$$

and

$$\gamma = \frac{1-2\nu}{2(1-\nu)} = \frac{\beta^2}{\alpha^2} \ . \tag{2}$$

The phase velocity of Rayleigh waves is denoted by c, and α, β are the velocities of dilatational (P) and shear (S) waves, respectively. For convenience we briefly summarise the results by Malischewsky (2000). By introducing the auxiliary functions

$$h_1(\gamma)=3\sqrt{33-186\gamma+321\gamma^2-192\gamma^3},$$
$$f(x,\gamma)=h_2(\gamma)=-17+45\gamma+h_1(\gamma),\ h_3(\gamma)=17-45\gamma+h_1(\gamma),$$
$$h_4(\gamma)=-\gamma+1/6, \tag{3}$$

we may write the three roots of (1) as

$$x(\gamma)=\frac{2}{3}\left[4-\sqrt[3]{h_3(\gamma)}+\operatorname{sign}\left[h_4(\gamma)\right]\cdot\sqrt[3]{\operatorname{sign}\left[h_4(\gamma)\right]\cdot h_2(\gamma)}\right], \tag{4}$$

$$x_c(\gamma)=\frac{1}{3}\left[8+\left(1\pm \mathrm{i}\sqrt{3}\right)\sqrt[3]{h_3(\gamma)}+\left(-1\pm \mathrm{i}\sqrt{3}\right)\right.$$
$$\left.\cdot\operatorname{sign}\left[h_4(\gamma)\right]\cdot\sqrt[3]{\operatorname{sign}\left[h_4(\gamma)\right]\cdot h_2(\gamma)}\right]. \tag{5}$$

We assume that the cubic roots are located in the first and fourth quadrants, depending on the sign of the imaginary part under the root. This choice is carried out automatically by software such as MATHEMATICA or FORTRAN, so that no special difficulties arise. Eq (4) yields the Rayleigh wave, while the two complex roots are obtained from (5) in the range $\nu_0 < \nu \le 0.5$, where ν_0=0.26308 … is a critical Poisson ratio, which was first obtained analytically by Malischewsky (2000) as

$$\nu_0=\frac{1}{24}\left[4-55\sqrt[3]{\frac{2}{N}}+\sqrt[3]{4N}\right], N=97+57\sqrt{\frac{57}{2}}\,. \tag{6}$$

The corresponding value of γ is γ_0 = 0.3215..., obtained by Rahman and Barber (1995) in a more complicated notation. It is found from

$$\gamma_0=\frac{1}{192}\left[107-\frac{455}{\sqrt[3]{N_1}}+\sqrt[3]{N_1}\right], N_1=-77293+7296\sqrt{114}\,. \tag{7}$$

Before proceeding to utilise these useful expressions, it may be appropriate to make some general remarks on the solutions of Rayleigh's equation. It is a well-known fact, which has been mathematically proved (see e. g. Narasimhan, 1993), that Rayleigh's equation has a solution x<1 for all admissible values of Poisson's ratio. The ratio c/β is a continuously increasing function of Poisson's ratio, which varies from 0.6889… for ν = -1 to 0.9553… for ν = 0.5 . The latter value deserves to be examined more closely. The value ν = 0.5 can mean either that the material is a liquid (β = 0), or that α is infinite, as for an incompressible material such as rubber. The first case may be excluded here, as Rayleigh waves cannot propagate in liquids. Ewing *et al.* (1957) proposed a simplified Rayleigh equation for incompressible materials

$$f_{inc}(x)=x^3-8x^2+24x-16=0, \tag{8}$$

whose analytical solution follows easily from our general solution (4):

$$x_{inc}=\frac{2}{3}\left(4+\sqrt[3]{-17+3\sqrt{33}}-\sqrt[3]{17+3\sqrt{33}}\right)=0.9126...\,. \tag{9}$$

The complex roots for $\nu>\nu_0$ were described by Hayes and Rivlin (1962) as extraneous roots which arise because of the rationalisation of the original Rayleigh equation. They concluded that the complex roots lead to inadmissible displacement fields at infinity. Thus they attempted to extend the original result by Rayleigh (1885), where the same conclusion was reached for incompressible materials only. On the other hand, these additional roots do correspond to solutions of the original partial differential equation, and we should not jump to conclusions.

Undoubtedly these complex roots must be closely connected with the leaking modes of waveguides, a familiar phenomenon in seismology. These modes may be unphysical, but following Kamel and Felsen (1981) they may be regarded as accounting more efficiently for the continuous spectrum contribution of the normal mode within a *bounded part* of the waveguide. This remark concerns the cross-sectional coordinate of the waveguide. On the other hand, complex x-values lead to complex wavenumbers concerning the axial coordinate of the waveguide, so that these solutions cannot be extended in an axial direction from -∞ to +∞. Yet they could easily exist in the form of so-called evanescent waves, in the neighbourhood of lateral heterogeneities. Such a situation is far from exceptional in waveguide problems. In the context of seismology, *neighbourhood* can mean hundreds of kilometres.

Let us define

$$c_c/\beta=\sqrt{\operatorname{Re}\left[x_c\{\gamma(\nu)\}\right]} \tag{10}$$

and let us look more closely at these peculiar solutions with a "phase velocity" c_c. In doing this we keep in mind that there is no standard definition for the phase velocity of leaking modes. The ratio c_c/β is a continuously and weakly decreasing function of Poisson's ratio which varies from 1.89087… for $\nu = \nu_0$ to 1.8825… for ν = 0.5. Hence c_c is always greater than β, and it is even slightly greater than α in the range $\nu_0 < \nu <$ 0.3054... For ν_{crit} =0.3054... we have $c_c = \alpha$. Such unusual values are hardly surprising, as Malischewsky (1985) obtained higher leaking modes of Love waves with phase velocities exceeding 200% of the value of α in the half-space.

Let us go one step further. If we interpret c_c as the apparent velocity of incident or outgoing P and S waves belonging to the continuous spectrum

$$\sin\vartheta_P^{(c)}=\frac{\beta}{c_c\sqrt{\gamma}},\ \sin\vartheta_S^{(c)}=\frac{\beta}{c_c}, \tag{11}$$

the angles ϑ are with the normal at the surface. Then we may confirm (Figure 1) that such an interpretation is possible for S waves over the entire range $\nu_0 < \nu \le 0.5$, but for P waves it obtains only in the range $\nu_0 < \nu < \nu_{crit}$. Thus the following remarks may be pertinent. When surface waves encounter a

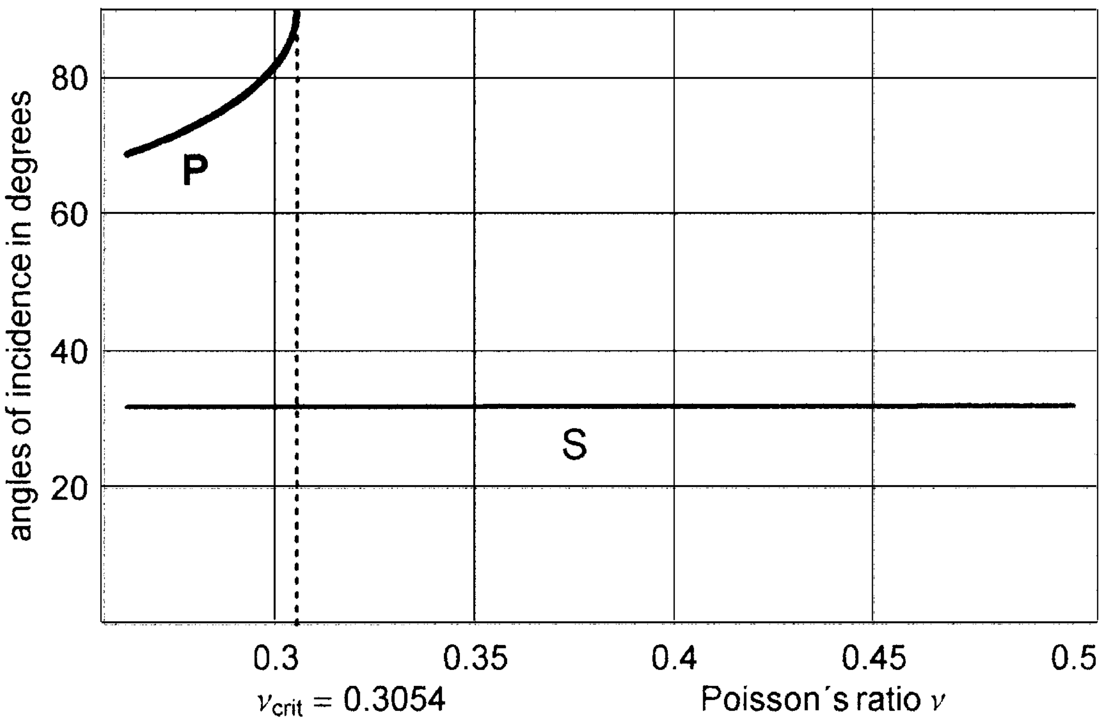

Fig. 1. Incidence angles versus ν of P (heavy) and S (light) waves corresponding to the apparent velocity derived from complex Rayleigh roots

lateral inhomogeneity they may generate an outgoing radiation field. On the other hand, the radiating modes (or body waves) form a standing wave field in the transverse direction. The conception of *radiating modes* originates from the physics of optical waveguides though it has been successfully adopted in seismology (see e. g. Malischewsky, 1987, and Maupin, 1996). The fact that modes with a standing wave pattern can produce an outgoing wave field is surprising but well established (Marcuse, 1974). The explanation is as follows. If radiation is excited by an imperfection in the waveguide it excites infinitely many radiation modes, which are superposed in such a way that the incoming parts of the standing wave cancel by constructive interference.

Thus it cannot be excluded that the "forbidden" complex solutions of Rayleigh's equation may be helpful in explaining the fine structure of seismograms near lateral disturbances. However, it is too early to draw any final conclusions.

In the following we concentrate on the range $\nu < \nu_0$ and especially on the solution of (1) at the critical points ν_0 and γ_0. It is a known fact that for $\nu \le \nu_0$ the solutions (5) become real and are related to certain critical reflections of longitudinal or transverse waves at a stress-free surface. We shall return to this soon, but first let us rewrite Eq (5) for real roots as follows:

$$\bar{x}_{1,2}(\gamma) = \frac{1}{3}\left(8 + 2\left\{\mathrm{Re}\left[\sqrt[3]{h_3(\gamma)}\right] \mp \sqrt{3}\cdot \mathrm{Im}\left[\sqrt[3]{h_3(\gamma)}\right]\right\}\right)$$
$$\text{for } \gamma_0 < \gamma \le 0.75 \quad . \tag{12}$$

According to (4) the root x_0 (the Rayleigh-wave root) has no peculiarities for ν_0 or γ_0, since

$$x_0 = \frac{4}{3}(2-\tilde{\nu}) = \frac{4}{3}(2-\tilde{\gamma}), \text{ with } \tilde{\nu} = \sqrt{\frac{10\nu_0 - 4}{\nu_0 - 1}} \text{ and } \tilde{\gamma} = \sqrt{12\gamma_0 - 2}, \tag{13}$$

but the root x_{c0} that follows from (5) for $\nu = \nu_0$,

$$x_{c0} = \frac{4}{3}\left(2 + \frac{1}{2}\tilde{\nu}\right) = \frac{4}{3}\left(2 + \frac{1}{2}\tilde{\gamma}\right), \tag{14}$$

is more interesting because of its significance as a double root of the cubic equation (1). Rahman and Barber (1995) provide solutions for (13) and (14) that are more complicated, as they contain cubic roots which may now be avoided.

THE CONNECTION WITH BODY WAVE REFLECTIONS

At a stress-free surface let us denote the angles of incidence with the vertical by ϑ_P and ϑ_S for longitudinal and transverse waves, respectively. Normally an incident wave will produce both kinds of elastic waves, so that it becomes necessary to consider the four reflection coefficients $R_{PP}(\vartheta_P)$, $R_{PS}(\vartheta_S)$, $R_{SS}(\vartheta_S)$, $R_{SP}(\vartheta_S)$. We adopt here the seismological notation for longitudinal and transverse waves as P- and S-waves, respectively. By setting $R_{PP}(\vartheta_P)=0$ or $R_{SS}(\vartheta_S)=0$ we obtain the Brewster angles θ_P and θ_S, defined as those angles at which an incident P-wave produces an S-wave only and vice versa.

It is an interesting though well-known fact (see e. g. Ewing *et al*., (1957), that both the equations $R_{PP}(\vartheta_P)=0$ and $R_{SS}(\vartheta_S)$ can be transformed into the cubic equation (1) by defining

$$x=\frac{1}{\gamma\sin^2\theta_P}\text{ for P-waves and }x=\frac{1}{\sin^2\theta_S}\text{ for S-waves}. \quad (15)$$

This connection between body wave reflection coefficients and Rayleigh waves becomes more transparent when we realise that the latter may be considered as a system of simultaneously incident P and S waves at a free surface with complex angles of incidence.

Now the root x can be understood as the ratio c_a^2/β^2, where c_a is the apparent velocity at the surface of P- and S-waves, respectively. This may be easily checked by inspecting the well-known expressions for R_{PP} and R_{SS} (see e. g. Achenbach, 1973). We present here for the first time a 3D graph of these coefficients in the range $-1\leq\nu\leq 0.5$ (Figure 2). Their intersection curves $\theta_P(\nu)$ and $\theta_S(\nu)$ with the plane $z = 0$ (blue) were first published by Arenberg (1948).

Usually there are two pairs of values θ_{P1}, θ_{P2} and θ_{S1}, θ_{S2}, for each value of ν. For $\nu = 0.25$ we obtain the following simple expressions:

$$\sin\theta_{P1}=\sqrt{3/2}\rightarrow\theta_{P1}=60°;\sin\theta_{P2}=3/\sqrt{6+2\sqrt{3}}\rightarrow\theta_{P2}=77.21°,$$
$$\sin\theta_{S1}=1/2\rightarrow\theta_{S1}=30°;\sin\theta_{S2}=1/\sqrt{2+2/\sqrt{3}}\rightarrow\theta_{S2}=34.26°. \quad (16)$$

However, for $\nu=\nu_0$ there is only one Brewster angle, namely θ_{P0} = 68.86° for P-waves and θ_{S0} = 31.93° for S-waves. These values may be simply related to ν_0 or γ_0 by way of Eqs (14) and (15).

$$\sin\theta_{P0}=\sqrt{\frac{3(\nu_0-1)}{(2\nu_0-1)(4+\tilde{\nu})}}=\sqrt{\frac{3}{2\gamma_0(4+\tilde{\gamma})}}, \quad (17)$$

$$\sin\theta_{S0}=\sqrt{\frac{3/2}{4+\tilde{\nu}}}=\sqrt{\frac{3/2}{4+\tilde{\gamma}}}. \quad (18)$$

Note, that the curve $\theta_{P2}(\nu)$ has a maximum of 90° at $\nu\rightarrow 0$. However, this value is to be excluded, because it requires a separate consideration in connection with the so-called Goodier-Bishop waves (see e. g. Ewing *et al.*, 1957). These strange inhomogeneous waves feature a linear amplitude-depth dependence. They are necessary to meet the boundary conditions at the free surface of a half-space at grazing incidence ($\theta = 90°$) for P or S waves. Goodier-Bishop waves are inadmissible at infinity, but Malischewsky (1971) showed that they can exist within bounded layers of a layered half-space.

It should be mentioned that Brewster angles are always of great significance in seismology. Thus these results are applicable to the problem of change of polarity of seismic signals such as *pP*, *PP*, *sS*, and *SS* after reflection at the Earth´s surface, and they may be used for the investigation of crustal structure with such phases (see e. g. Papazachos, 1964).

PECULIARITIES OF R_{PS} AND R_{SP}

After obtaining Brewster's angles θ_P and θ_S by equating R_{PP} and R_{SS} to zero, it may be interesting to investigate the corresponding associated coefficients R_{PS} and R_{SP} for any peculiarities they may offer. Such peculiarities indeed exist as Figure 3 shows. Thus R_{PS} has a maximum at $\vartheta_P^{(m)}$ =40.37° and ν_m = 0.1481 and R_{SP} has a saddle point at $\vartheta_S^{(s)}$=27.84° and ν_s = 0.0043. These features are presented here for the first time. Clearly they are not connected with the special points (17) and (18).

Note that, because of its vicinity to zero, the saddle point at ν_s would be hard to discover unless negative Poisson's ratios are considered. Finally, let us mention that there are intersection points between all combinations of the reflection coefficients. The peculiar features of these points will be discussed in a forthcoming paper.

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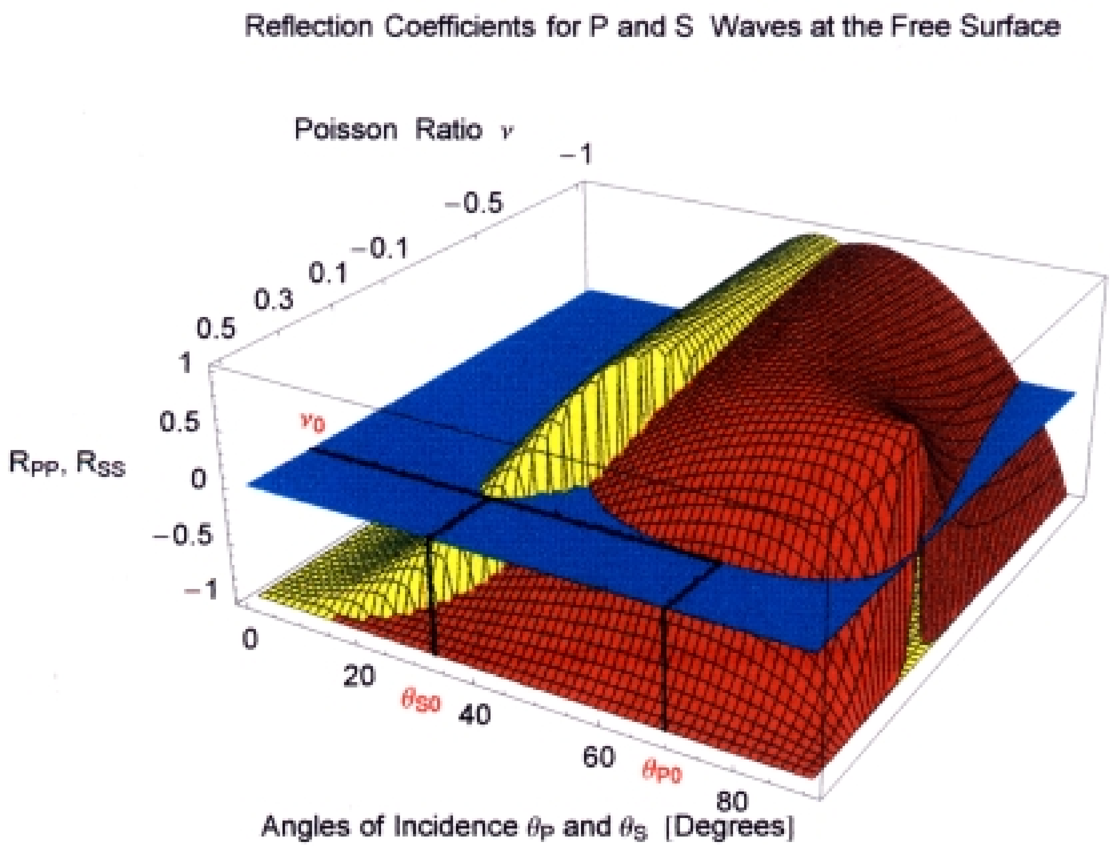

Fig. 2. The reflection coefficients R_{PP} (red) and R_{SS} (yellow) as functions of the Poisson ratio and the angle of incidence

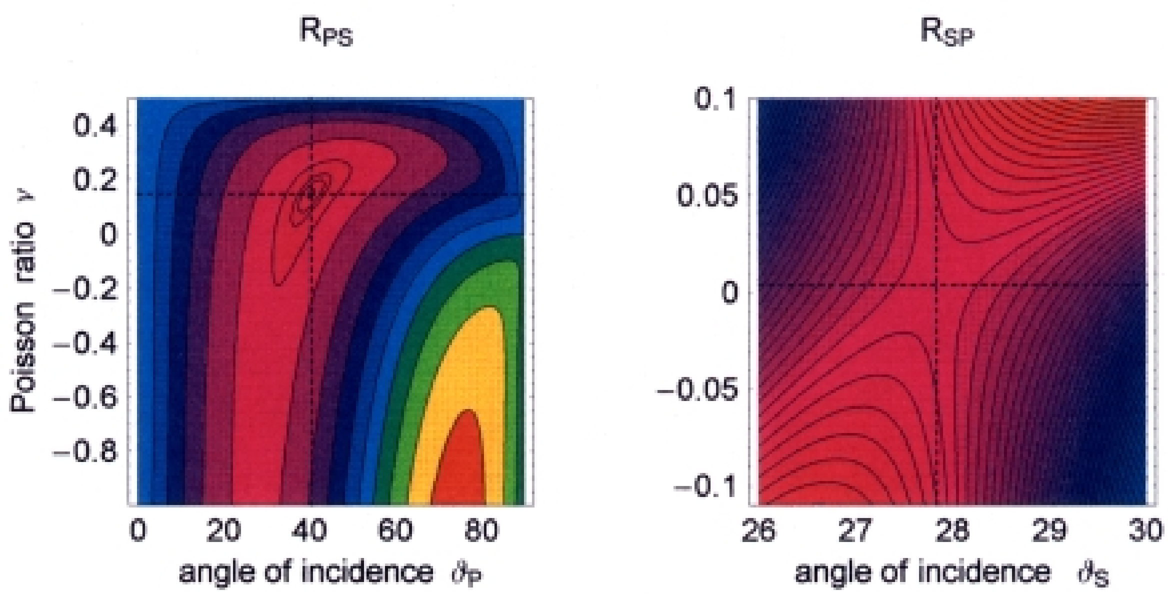

Fig. 3. Contour plots for the reflection coefficients R_{PS} and R_{SP} as functions of the Poisson ratio and the angle of incidence. For better visibility of the saddle point the presentation for R_{SP} is zoomed.

BIBLIOGRAPHY

ACHENBACH, J. D., 1984. Wave Propagation in Elastic Solids, Elsevier, New York.

ARENBERG, D. L., 1948. Ultrasonic Solid Delay Lines, *J. Acoust. Soc. America*, *20*, 1-26.

EWING, W. M., W. S. JARDETZKY and F. PRESS, 1957. Elastic Waves in Layered Media, Mc Graw-Hill Inc., New York.

HAYES, M. and R. S. RIVLIN, 1962. A Note on the Secular Equation for Rayleigh Waves. *Z. Angew. Math. Phys.*, *13*, 80-83.

KAMEL, A. and L. B. FELSEN, 1981. Hybrid Ray-Mode Formulation of SH-Motion in a Two-Layer Half-Space. *Bull. Seism. Soc. Am.*, *71*, 1763-1781.

LAKES, R. S., 1987. Negative Poisson's Ratio Materials. *Science*, *238*, 551.

MALISCHEWSKY, P., 1971. Consideration of Certain Singularities in Haskell's Matrix Method. *Gerl. Beitr. Geophys.*, *80*, 457-462.

MALISCHEWSKY, P., 1985. A Semi-Analytical Method for the Calculation of Leaking Love-Wave Modes. *Wave Motion*, *7*, 253-262.

MALISCHEWSKY, P., 1987. Surface Waves and Discontinuities, Elsevier, Amsterdam.

MALISCHEWSKY, P. G., 2000. Comment to "A New Formula for the Velocity of Rayleigh Waves" by D. Nkemzi. *Wave Motion*, *31*, 93-96.

MARCUSE, D., 1974. Theory of Dielectric Optical Waveguides, Academic Press, New York.

MAUPIN, V., 1996. The Radiation Modes of a Vertically Varying Half-Space: a New Representation of the Complete Green's Function in Terms of Modes. *Geophys. J. Int.*, *126*, 762-780.

NARASIMHAN, M. N. L., 1993. Principles of Continuum Mechanics, John Wiley & Sons, New York.

NKEMZI, D., 1997. A New Formula for the Velocity of Rayleigh Waves. *Wave Motion*, *26*, 199-205.

PAPAZACHOS, B., 1964. Angle of Incidence and Amplitude Ratio of P and PP Waves. *Bull. Seism. Soc. Am.*, *54*, 105-121.

RAHMAN, M. and J. R. BARBER, 1995. Exact Expressions for the Roots of the Secular Equation for Rayleigh Waves. *ASME J. Appl. Mech*., *62*, 250-252.

RAYLEIGH, J. W. S., 1885. On Waves Propagating along the Plane Surface of an Elastic Solid. *Proc. London Math. Soc*., *17*, 4-11.

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