Short Note

A note on Rayleigh-wave velocities as a function of the material parameters

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RESUMEN

Se demuestra que la nueva fórmula de Nkemzi (2003) para la velocidad de las ondas de Rayleigh en un semiespacio homogéneo se deriva de la fórmula de Malischewsky (2000a) por medio de una transformación algebraica simple. Una transformación similar es también llevada a cabo para las raíces complejas.

PALABRAS CLAVE: Velocidad de fase de las ondas Rayleigh, parámetros materiales.

ABSTRACT

A new formula was proposed by Nkemzi (2003) for the Rayleigh-wave velocity in a homogeneous half-space. This formula may be derived from the earlier formula by Malischewsky (2000a) by a simple algebraic transformation. A similar transformation may be carried out for the complex roots as well.

KEY WORDS: Phase velocity of Rayleigh waves, material parameters.

INTRODUCTION

The Rayleigh-wave velocity c is a fundamental quantity which interests researchers in seismology and geophysics, and in other fields of physics and the material sciences. In the homogeneous halfspace, c does not depend on frequency as it does in a layered half-space. It depends only on the velocities α of longitudinal P waves and β of shear S waves. For many purposes it is useful to have a simple formula at hand, which expresses the dependence of c on the material parameters.

The recent history of this development is as follows. Setting $x = (c/\beta)^2$ we may write the well-known Rayleigh equation

$$x^3 - 8x^2 + 8x(3-2\gamma) - 16(1-\gamma) = 0, \qquad (1)$$

where

$$\gamma = \frac{1-2v}{2(1-v)} = \left(\frac{\beta}{\alpha}\right)^2, \qquad (2)$$

and v is Poisson's ratio. From the theory of functions Nkemzi (1997) wrote the solution of (1) for Rayleigh waves in the form of an integral representation:

$$x = 4 - 4\gamma^{1/2} - \frac{1}{\pi}\int_1^{1/\gamma} \arctan\left\{\frac{4(1-\gamma t)^{1/2}(t-1)^{1/2}}{(2-t)^2}\right\} dt.$$

$$(3)$$

By applying the Wiener-Hopf technique Brock (1998) obtained a similar integral representation for the Rayleigh-wave root and derived additionally a more complicated one for the Stoneley-wave root. Romeo (2001) confirmed the correctness of representation (3). However, Equation (3) is cumbersome for numerical applications, as is the formula derived from it by Nkemzi (1997). The latter formula is incorrect as pointed out by Malischewsky (2000a). A more practical and correct version was provided by Malischewsky (2000a, 2000b), which was comprehensively discussed and confirmed by Pham and Ogden (2004). These authors presented also a new formula, which is not to be discussed here. Recently, Nkemzi (2003) has shown that the useful and simple formula

$$x = \frac{2}{3}\left[4 - \sqrt[3]{h_3(\gamma)} + \frac{2(1 - 6\gamma)}{\sqrt[3]{h_3(\gamma)}} \right], \qquad (4)$$

may be derived from the above integral representation (3). In this note we show that Equation (4) is essentially identical with Malischewsky's formula as it may be derived from it by means of a simple transformation.

A DERIVATION OF RAYLEIGH-WAVE VELOCITIES

Let us summarize the procedure in Malischewsky (2000a, 2000b). From the theory of cubic equations,

$$h_1(\gamma) = 3\sqrt{33 - 186\gamma + 321\gamma^2 - 192\gamma^3},$$
$$h_2(\gamma) = -17 + 45\gamma + h_1(\gamma), \qquad (5)$$
$$h_3(\gamma) = 17 - 45\gamma + h_1(\gamma), \quad h_4(\gamma) = 1/6 - \gamma,$$

where h_1, h_2, h_3 and h_4 are auxiliary functions and the square root is taken as positive. Then the solution of the Rayleigh equation may be written

$$x(\gamma) = \frac{2}{3}\left[4 - \sqrt[3]{h_3(\gamma)} + \mathrm{sign}[h_4(\gamma)] \cdot \sqrt[3]{\mathrm{sign}[h_4(\gamma)] \cdot h_2(\gamma)} \right], \qquad (6)$$

where the main values of the cubic roots are to be used. The introduction of the sign function is necessary because of the root of $h_2(\gamma)$ at $\gamma = 1/6$. Note that this solution has no irrational denominator as does equation (4).

Using the algebraic identities

$$h_2 \cdot h_3 \equiv 12^3 \cdot h_4^3 \qquad (7)$$

and

$$h_4 \equiv \mathrm{sign}[h_4] \cdot |h_4|, \qquad (8)$$

we find that

$$h_2 = \frac{12^3 [\mathrm{sign}(h_4) \cdot |h_4|]^3}{h_3}, \qquad (9)$$

so that the third term of (6) within the brackets becomes

$$12 \frac{\mathrm{sign}(h_4) \sqrt[3]{[\mathrm{sign}(h_4)]^4 \cdot |h_4|^3}}{\sqrt[3]{h_3}} = \frac{12 h_4}{\sqrt[3]{h_3}}.$$

Thus we may write (6) as

$$x(\gamma) = \frac{2}{3}\left[4 - \sqrt[3]{h_3(\gamma)} + \frac{2(1 - 6\gamma)}{\sqrt[3]{h_3(\gamma)}} \right], \qquad (10)$$

which is identical with Eq (4) after Nkemzi (2003). If we compare the two versions (6) and (10), expression (10) has the advantage of not containing the sign function; on the other hand, expression (6) does not contain an irrational denominator. It should be noted that such simple closed formulas cannot be obtained for Stoneley waves, because their secular equation is a more complicated irrational equation, whose reduction to a polynom is not straightforward.

The complex roots x_c of (1) as given by Malischewsky (2000a, 2000b) may be transformed in the same manner, i.e.

$$x_c(\gamma) = \frac{1}{3}\left[8 + (1 \pm i\sqrt{3}) \sqrt[3]{h_3(\gamma)} + \frac{2(-1 \pm i\sqrt{3})(1 - 6\gamma)}{\sqrt[3]{h_3(\gamma)}} \right]. \qquad (11)$$

Here γ varies over the range $0 \le \gamma \le \gamma_0$, where $\gamma_0 = 0.3215\ldots$. These complex roots can become important when calculating complete theoretical seismograms (e.g., Schröder and Scott, 2001; Harris and Achenbach, 2002). The functions x and x_c were shown graphically in Malischewsky (2000a).

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